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**Stochastic homogenization of elliptic equation and optimal control**

by

Zhen Li

A dissertation submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:

Steven L. Hou, Major Professor

Scott Hansen

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Iowa State University

Ames, Iowa

2012

## DEDICATION

I would like to dedicate this thesis to my wife Yanfei Wang and to my daughter Carolyn without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance and financial assistance during the writing of this work.

## TABLE OF CONTENTS

<b>LIST OF TABLES</b> . . . . .	v
<b>LIST OF FIGURES</b> . . . . .	vi
<b>ACKNOWLEDGEMENTS</b> . . . . .	vii
<b>ABSTRACT</b> . . . . .	viii
<b>CHAPTER 1. Overview</b> . . . . .	1
1.1 Introduction . . . . .	1
1.2 One Dimensional Case . . . . .	2
1.3 Stochastic Homogenization for Elliptic Equations with Ergodic Assumption . .	4
1.4 Wiener Chaos Expansion for Homogenization of Stochastic Elliptic Equation .	5
<b>CHAPTER 2. One Dimension Case</b> . . . . .	7
2.1 Introduction . . . . .	7
2.2 One Dimension Case . . . . .	8
2.3 Optimal Control Problem and Partial Differential Equation . . . . .	9
2.4 Numerical Example . . . . .	11
2.4.1 Discretization of the system . . . . .	11
2.4.2 Comparison of the results for $a_\epsilon$ and $a_0$ . . . . .	13
2.4.3 Comparison of values of objective functions . . . . .	14
<b>CHAPTER 3. Stochastic Homogenization for Elliptic Equations with Er-</b> <b>godic assumption</b> . . . . .	21
3.1 Introduction . . . . .	21
3.2 Birkhoff Ergodic Theorem . . . . .	22

3.3 Numerical Example . . . . .	26
<b>CHAPTER 4. Wiener Chaos Expansion for Stochastic Elliptic Equation and</b>	
<b>Optimal Control . . . . .</b>	<b>31</b>
4.1 Wiener Chaos Expansion and Hermite Polynomials . . . . .	31
4.1.1 Brocher-Minlos Theorem . . . . .	31
4.1.2 Hermite Polynomials . . . . .	32
4.1.3 Wiener Chaos Expansion . . . . .	35
4.2 Kondratiev Spaces and Wick Products . . . . .	36
4.3 Optimal control of stochastic elliptic equation . . . . .	39
4.4 Numerical Example . . . . .	45
4.4.1 Discretization of the control problem . . . . .	45
4.4.2 Numerical example . . . . .	48
<b>BIBLIOGRAPHY . . . . .</b>	<b>52</b>

## LIST OF TABLES

Table 2.1	Comparison of values of objective functions . . . . .	15
Table 4.1	Values of objective function $J$ for each iteration . . . . .	50
4.2	Summary of objective function . . . . .	51

## LIST OF FIGURES

Figure 2.1	The graph of $a_\epsilon(x)$ with $\epsilon = 1, \epsilon = 0.5$ and $\epsilon = 0.05$ . . . . .	12
Figure 2.2	The graph of $u_\epsilon$ and $U(x) = \sin(2\pi x), 0 \leq x \leq 1, \epsilon = 0.01$ . . . . .	13
Figure 2.3	The graph of $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.01$ . . . . .	14
Figure 2.4	The graph of $u_\epsilon$ and $U(x) = \sin(2\pi x), 0 \leq x \leq 1, \epsilon = 0.001$ . . . . .	15
Figure 2.5	The graph of $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.001$ . . . . .	16
Figure 2.6	The graph of $u_\epsilon$ and $u_0, \epsilon = 0.01$ . . . . .	16
Figure 2.7	The graph of error between $u_\epsilon$ and $u_0, \epsilon = 0.01$ . . . . .	17
Figure 2.8	The graph of $v_\epsilon$ and $v_0, \epsilon = 0.01$ . . . . .	17
Figure 2.9	The graph of error between $v_\epsilon$ and $v_0, \epsilon = 0.01$ . . . . .	18
Figure 2.10	The graph of $u_\epsilon$ and $u_0, \epsilon = 0.001$ . . . . .	18
Figure 2.11	The graph of error between $u_\epsilon$ and $u_0, \epsilon = 0.001$ . . . . .	19
Figure 2.12	The graph of $v_\epsilon$ and $v_0, \epsilon = 0.001$ . . . . .	19
Figure 2.13	The graph of error between $v_\epsilon$ and $v_0, \epsilon = 0.001$ . . . . .	20
Figure 3.1	The graph of $a_\epsilon^T(x, \omega)$ with $\epsilon = 1$ . . . . .	29
Figure 3.2	The graph of $a_\epsilon^T(x, \omega)$ with $\epsilon = 0.5$ . . . . .	29
Figure 3.3	The graph of $a_\epsilon^T(x, \omega)$ with $\epsilon = 0.05$ . . . . .	30
Figure 3.4	The graph of $\bar{u}(x)$ and target function $U(x) = \sin(2\pi x)$ . . . . .	30
Figure 4.1	The graph of $a_\epsilon(x, \omega)$ with $\epsilon = 1, \epsilon = 0.25$ and $\epsilon = 0.05$ . The smaller the $\epsilon$ is, the more oscillated the coefficient is. . . . .	49
Figure 4.2	The graph of the convergence of values of objective function $J$ . . . . .	50
Figure 4.3	The graph of WCE solution $u(x, \omega)$ under control, its mean value $E[u(x, \omega)]$ and the target function $U(x)$ . . . . .	51

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## ABSTRACT

In this thesis, we mainly study the numerical methods for stochastic homogenization of elliptic optimal control problem, where there is random variable involved in the constraint. We start with a simple one-dimensional optimal control problem and derive the effective equations of the original optimal control problem and the theory results regarding the convergence of solutions have been studied in [18]. Then an elliptic optimal control problem with coefficient having ergodicity is studied and convergence theorem is also given regarding the effective equations. Finally, another elliptic optimal control problem, where the normal product is replaced by the wick product, is discussed. An algorithm used to search the optimal solution is obtained. Numerical examples are given in each chapter.

## CHAPTER 1. Overview

### 1.1 Introduction

In the recent years, there has been an increased interest in the stochastic partial differential equations (SPDEs) and SPDEs are known to be an effective tool in modeling complex physical and engineering phenomena. The reason for increasing interest in uncertainties is that uncertainties remain in most models of real world problems. Uncertainties arise either due to insufficient or lack of knowledge (epistemic uncertainties), or due to the intrinsic variabilities of physical quantities (aleatoric uncertainties), e.g. due to heterogeneities in materials. Due to the intrinsic heterogeneities of materials, homogenization is also widely applied in obtaining the corresponding effective equations. It allows us to study the global behavior of the materials with heterogeneities and describe the macroscopic behavior of the systems with a fine microstructure.

Because of the uncertain terms (random term) involved, the numerical simulation becomes hard and sometime probability measure theory is also needed. But, we do have couple ways to quantify the uncertainties (random terms) [8], [15], [16], [24] and [33].

- Stochastic modeling: Uncertainties (random term) are usually modeled by stochastic models. Stochastic models are widely used to describe the uncertainties, whose uncertainties, uncertain parameters, uncertain functions and domain are described by random variables, stochastic processes and random fields. We will focus on this method in this thesis.
- Fuzzy sets may be used to describe uncertainties. They describe the uncertainties by possibility functions specifying their degree of belonging to a set.

- Besides the stochastic model and fuzzy sets, set methods are independent of probability and possibility measure.

In this thesis, problems involving stochastic modeling, homogenization optimal control will be discussed and analyzed. Optimal control focus on the problem of finding the optimized solution for a system modeled by a set of differential equations (constraints) and a cost function (objective function) that will be minimized with such optimized solution. For the optimal control, adjoint equation-based methods are in common used for the solution of flow control and optimization problems in [11], [12].

For the stochastic modeling describing random term with homogenization and optimization involved, numerical simulation plays a important role in studying such problems. For this reason, there are many numerical methods have been developed for simulating SPDEs, such as Monte Carlo(MC) method in [25].

## 1.2 One Dimensional Case

First, one dimensional case is taken for example and discussed for the homogenization of an optimal control problem in which the state equation (given by a second-order elliptic boundary value problem) has rapidly oscillating coefficients. This work can be found in [6], [18] and [29]. Let  $f \in L^2(\Omega)$ ,  $A$  and  $B$  are matrices whose entries are functions on bounded domain  $\Omega$  with smooth boundary.  $B$  is also symmetric and nonnegative.  $N > 0$  is a given constant. Let  $\theta(x)$  be a control variable and the optimal control problem is defined as follows,

$$\begin{cases} -\operatorname{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the state  $u = u(\theta)$  is thus defined as the weak solution in  $H_0^1(\Omega)$  of above problem.

Then the cost function is given by

$$J(\theta) = \frac{1}{2} \int_{\Omega} (B\nabla u, \nabla u) \, dx + \frac{N}{2} \int_{\Omega} \theta^2(x) \, dx.$$

Minimization of the above cost function is a standard minimization problem, a discussion of which can be found in the book by [21]. A reduced form is obtained by introducing a new

adjoint state  $p$ ,

$$\begin{cases} -\operatorname{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A^t\nabla p - B\nabla u) = 0 & \text{in } \Omega, \end{cases}$$

where  $u, p \in H_0^1(\Omega)$ , and the optimal control  $\theta^*$  can be characterized by such inequality

$$\int_{\Omega} (p + N\theta^*)(\theta - \theta^*) dx \geq 0 \quad \forall \theta \in S,$$

where  $S$  is a subset of  $L^2(\Omega)$ .

From the theorems and corollary from [18], one can have

$$\theta_{\epsilon}^* \rightharpoonup \theta^* \quad \text{weakly in } L^2(\Omega),$$

where  $\theta^*$  is also an optimal control defined by a problem of the same type with matrices  $A^*$  and  $B^*$ . More convergence results can also be found in [18]. With Lagrange multiplier, the equivalent equation of above optimal control problem is

$$\begin{cases} -\operatorname{div}(A_{\epsilon}\nabla u_{\epsilon}) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A_{\epsilon}^t\nabla p_{\epsilon} - B\nabla u_{\epsilon}) = 0 & \text{in } \Omega, \\ u_{\epsilon} = p_{\epsilon} = 0 & \text{on } \partial\Omega, \end{cases}$$

and based on the convergence theorem in [18], a system of effective equations is shown below,

$$\begin{cases} -\operatorname{div}(A_0\nabla u_0) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A_0^t\nabla p_0 - B\nabla u_0) = 0 & \text{in } \Omega. \end{cases}$$

The convergence results are

$$u_{\epsilon} \rightharpoonup u_0, \text{ as } \epsilon \rightarrow 0 \tag{1.2.1}$$

$$p_{\epsilon} \rightharpoonup p_0, \text{ as } \epsilon \rightarrow 0. \tag{1.2.2}$$

A numerical example is also given to shows the weak convergence between  $u_{\epsilon}$  and  $u_0$ , as defined above.

### 1.3 Stochastic Homogenization for Elliptic Equations with Ergodic Assumption

For some stochastic elliptic equations with random coefficients, the coefficients may have the ergodic property. Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems, and studies long-term behavior in dynamical systems from a statistical point of view. Its initial development was motivated by problems of statistical physics. Recently, there is increasing interest in stochastic homogenization with stationary ergodic random variable over a probability space, and those work can be found in [1], [3], [19], [27], [28] and [30].

For the coefficients  $a(x, \omega)$  with ergodic property, we assume that the ergodic process is involved, and

$$a^T(x, \omega) = a(T(x)\omega), \quad (1.3.1)$$

where  $a(x, \omega)$  is a fixed random variable and  $T = T(x) : \Omega \rightarrow \Omega$  is a transformation which preserves the measure  $\mu$  on  $\Omega$ .  $T(x) : \Omega \rightarrow \Omega$ ,  $x \in \mathbb{R}^m$  is called a dynamical system with  $m$ -dimensional time, or simply an  $m$ -dimensional dynamical system if it satisfies conditions:  $T(0)$  is a identity mapping;  $T$  is measure preserving on  $\Omega$ ; and  $f(T(x)\omega)$  is measurable. Under these ergodicity assumptions, an effective equation involving deterministic coefficients is obtained.

The solution of

$$\begin{cases} - (a_\varepsilon^T(x, \omega) u'_\varepsilon(x, \omega))' = f(x) & x \in L = (0, 1), \\ u_\varepsilon(0, \omega) = u_\varepsilon(1, \omega) = 0. \end{cases}$$

converges to the solution of

$$\begin{cases} - \frac{d}{dx} \left( \bar{a}(x) \frac{d}{dx} \bar{u}(x) \right) = f(x) & x \in \Omega = (0, 1), \\ \bar{u}(0) = \bar{u}(1) = 0. \end{cases}$$

in  $L^2(0, 1)$ , where  $\bar{a} = \frac{1}{\int \frac{1}{a(x)} dx}$ .

## 1.4 Wiener Chaos Expansion for Homogenization of Stochastic Elliptic Equation

Stochastic elliptic models (with random coefficients) has been widely used in modeling the physical and engineering problems [13], [17], such as reservoir and groundwater simulations. It makes more sense to assume that the permeability of porous media as a spatial random process instead of a deterministic one. There are two versions stochastic models

$$-\nabla(a(x, \omega)\nabla u(x)) = f(x) + \theta(x) \quad (1.4.1)$$

$$-\nabla(a(x, \omega) \diamond \nabla u(x)) = f(x) + \theta(x), \quad (1.4.2)$$

where  $\omega$  implies randomness and  $\diamond$  denotes the Wick products.

For (1.4.1), a strong ellipticity condition is needed for wellposedness by Lax-Milgram lemma, i.e., the coefficient  $a(x, \omega)$  must be strictly positive bounded. However, because of randomness involved in the coefficient  $a(x, \omega)$ , the standard ellipticity condition may not hold. A general theory of bilinear SPDEs that includes, in particular, (1.4.2) was developed recently in [22], where the regular product is replaced by the Wick product. The Wick product is a regularization procedure to alleviate the singularity caused by the noise mentioned above. From the view of mathematics, it is a version of Malliavin divergence operator corresponding to the Itô-Skorohod integral [32]. For the model (1.4.2), Wiener chaos expansion is a very useful tool in numerical simulation.

Wiener chaos expansion (WCE), also referred to polynomial expansion, is a non-sampling based method to determine evolution of uncertainty in dynamical system, when there is probabilistic uncertainty in the system parameters. It is being more and more popular in the recent years in simulation of the stochastic partial differential equations (SPDEs) with random coefficients and/or random forcing term involved.

The WCE was first introduced by Wiener, where Hermite polynomials were used to model stochastic processes with Gaussian random variables. The WCE discussed in this dissertation is based on the version of Cameron and Martin [4], who developed a more explicit and intuitive formulation for the Wiener chaos expansion with Hermite polynomial involved. The main improvement is the discretization of the white noise process by Fourier expansion and it makes

the simulation of solution of SPEs be accessible because it is easy to obtain the numerical solution.

We are going to consider the Wiener Chaos solution of the stochastic elliptic equation (SEE)

$$\begin{cases} -(a_\epsilon(x, \omega) \diamond u'_\epsilon(x, \omega))' = f(x) + \theta(x) & x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0, \end{cases} \quad (1.4.3)$$

with the objective function

$$J_\epsilon(u, \theta) = \frac{c}{2} E \left[ \int_0^1 |u_\epsilon - U|^2 dx \right] + \frac{1}{2} \int_0^1 \theta^2(x) dx, \quad (1.4.4)$$

where  $c$  is a constant and  $U(x)$  is deterministic.

The work for the application of WCE in numerical methods for solving elliptic equations with random coefficients can be found in [9], [10] and [31]. But the optimal control problem based on stochastic elliptic equation with rand coefficient involved (1.4.3) and (1.4.4), is very challenging and difficult to simulate. There are very few works regarding the application of WCE in such problem and these works can be found in [23] and [26].

## CHAPTER 2. One Dimension Case

### 2.1 Introduction

In this chapter, we will discuss the homogenization of an optimal control problem in which the state equation (given by a second-order elliptic boundary value problem) has rapidly oscillating coefficients. We just consider the one-dimensional case in this thesis. Let  $f \in L^2(\Omega)$ ,  $A$  and  $B$  are matrices whose entries are functions on bounded domain  $\Omega$  with smooth boundary.  $B$  is also symmetric and nonnegative.  $N > 0$  is a given constant. Let  $\theta(x)$  be a control variable and the optimal control problem which can be found in the paper by [18] is defined as follows,

$$\begin{cases} -\operatorname{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the state  $u = u(\theta)$  is thus defined as the weak solution in  $H_0^1(\Omega)$  of above problem.

Then the cost function is given by

$$J(\theta) = \frac{1}{2} \int_{\Omega} (B\nabla u, \nabla u) \, dx + \frac{N}{2} \int_{\Omega} \theta^2(x) \, dx.$$

Minimization of the above cost function is a standard minimization problem, a discussion of which can be found in the book by [21] and we obtain a reduced form by introducing a new adjoint state  $p$ ,

$$\begin{cases} -\operatorname{div}(A\nabla u) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A^t \nabla p - B\nabla u) = 0 & \text{in } \Omega, \end{cases}$$

where  $u, p \in H_0^1(\Omega)$ , and the optimal control  $\theta^*$  can be characterized by such inequality

$$\int_{\Omega} (p + N\theta^*)(\theta - \theta^*) \, dx \geq 0 \quad \forall \theta \in S,$$



where  $S$  is a subset of  $L^2(\Omega)$ .

What we are interested in is that given a parameter  $\epsilon > 0$  which tends to zero, the matrices  $A$  and  $B$  above depend on  $\epsilon$ . And we also have the same assumptions on  $A_\epsilon$  and  $B_\epsilon$ . In Kesavan's paper, there are also following conclusions. Suppose  $A_\epsilon$  is matrix depending on  $\epsilon$ , then  $\theta_\epsilon^*$  exists and is bounded in  $L^2(\Omega)$ . Thus, we have

$$\theta_\epsilon^* \rightharpoonup \theta^* \quad \text{weakly in } L^2(\Omega),$$

where  $\theta^*$  is also an optimal control defined by a problem of the same type with matrices  $A^*$  and  $B^*$ . That paper also gives the following theorem. The solution  $(u_\epsilon, p_\epsilon)$  of system

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A_\epsilon^t \nabla p_\epsilon - B \nabla u_\epsilon) = 0 & \text{in } \Omega, \\ u_\epsilon = p_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is bounded and also have the following weak convergence result in  $(H_0^1(\Omega))^2$ ,

$$u_\epsilon \rightharpoonup u_0, \text{ as } \epsilon \rightarrow 0$$

$$p_\epsilon \rightharpoonup p_0, \text{ as } \epsilon \rightarrow 0$$

where  $u_0, p_0$  satisfy the following system of equations,

$$\begin{cases} -\operatorname{div}(A_0 \nabla u_0) = f(x) + \theta(x) & \text{in } \Omega \\ \operatorname{div}(A_0^t \nabla p_0 - B \nabla u_0) = 0 & \text{in } \Omega. \end{cases}$$

## 2.2 One Dimension Case

For the one-dimensional case,

$$\begin{cases} -\frac{d}{dx} \left( a_\epsilon \frac{du_\epsilon}{dx} \right) = f(x) + \theta(x) & \text{in } (0, 1), \\ \frac{d}{dx} \left( a_\epsilon \frac{dp_\epsilon}{dx} - b_\epsilon \frac{du_\epsilon}{dx} \right) = 0 & \text{in } (0, 1), \end{cases}$$

we also have the similar results. Suppose  $(0, 1) \subset \mathcal{R}$ , if  $\frac{1}{a_\epsilon} \rightharpoonup \frac{1}{a_0}$  weakly in  $L^\infty(0, 1)$  and  $b^* = \frac{a_0^2}{g_0}$

where

$$\frac{1}{g_0} = \frac{b_\epsilon}{a_\epsilon^2} \rightharpoonup \frac{1}{g_0} \text{ weakly in } L^\infty(0, 1).$$

Then we have the following weak convergence in  $H_0^1(0, 1)$ ,

$$u_\epsilon \rightharpoonup u_0, \text{ as } \epsilon \rightarrow 0$$

$$p_\epsilon \rightharpoonup p_0, \text{ as } \epsilon \rightarrow 0$$

where  $u_0, p_0$  satisfy the equations

$$\begin{cases} -\frac{d}{dx} \left( a_0 \frac{du_0}{dx} \right) = f(x) + \theta(x) & x \in (0, 1), \\ \frac{d}{dx} \left( a_0 \frac{dp_0}{dx} - b^* \frac{du_0}{dx} \right) = 0 & x \in (0, 1), \\ u_0 = p_0 = 0 & x = 0, 1. \end{cases}$$

In this thesis, we rewrite this optimal control problem and consider the following forms,

$$\begin{cases} -\frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) = f(x) + \theta(x) & x \in (0, 1), \\ u_\epsilon(x) = 0 & x = 0, 1. \end{cases}$$

And the cost function is

$$J(\theta) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx,$$

where  $\beta \gg 1$  and  $U(x)$  is a given function.  $a_\epsilon(x)$  is a function defined on  $[0, 1]$ . Thus, the optimal control  $\theta^*$  is the function in  $[0, 1]$  which minimizes  $J(\theta)$  for  $\theta(x) \in L^2(0, 1)$ .

## 2.3 Optimal Control Problem and Partial Differential Equation

It is difficult to solve this optimal problem directly. From the numerical analysis viewpoint, it is advantageous to convert this problem to an equivalent PDE problem. Then we are able to analyze it by finite element or finite difference methods on numerical analysis. Let's consider the following optimal control problem

$$\begin{cases} -\frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) = f(x) + \theta(x) & x \in (0, 1), \\ u_\epsilon(x) = 0 & x = 0, 1, \\ \min \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx. \end{cases}$$

Let  $v_\epsilon(x) \in L^2(0, 1)$  and  $v_\epsilon = 0$ , if  $x = 0, 1$ , then

$$\begin{aligned} L(u_\epsilon, \theta) &= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx \\ &= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx - \int_0^1 v_\epsilon \left( -\frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) - f(x) - \theta(x) \right) dx. \end{aligned}$$

By the integration by parts and  $v_\epsilon = 0, u_\epsilon = 0$  if  $x = 0, 1$ , we find that

$$\begin{aligned} \int_0^1 v_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) dx &= v_\epsilon a_\epsilon(x) \frac{du_\epsilon}{dx} \Big|_0^1 - \int_0^1 a_\epsilon(x) \frac{du_\epsilon}{dx} \frac{dv_\epsilon}{dx} dx \\ &= - \int_0^1 a_\epsilon(x) \frac{du_\epsilon}{dx} \frac{dv_\epsilon}{dx} dx \\ &= -u_\epsilon a_\epsilon(x) \frac{dv_\epsilon}{dx} \Big|_0^1 + \int_0^1 u_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) dx \\ &= \int_0^1 u_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} L(u_\epsilon, \theta) &= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx \\ &\quad + \int_0^1 v_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{du_\epsilon}{dx}) dx + \int_0^1 v_\epsilon f(x) dx + \int_0^1 v_\epsilon \theta(x) dx \\ &= \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx \\ &\quad + \int_0^1 u_\epsilon \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) dx + \int_0^1 v_\epsilon f(x) dx + \int_0^1 v_\epsilon \theta(x) dx. \end{aligned}$$

Then for any  $t(x), w(x) \in L^2(0, 1)$ , we should have

$$\begin{aligned} \langle \frac{\partial L}{\partial u_\epsilon}, w \rangle &= \beta \int_0^1 (u_\epsilon - U) w dx + \int_0^1 w \frac{d}{dx} (a_\epsilon(x) \frac{dv_\epsilon}{dx}) dx \\ &= \int_0^1 \left( \beta(u_\epsilon - U) + \frac{d}{dx} \left( a_\epsilon(x) \frac{dv_\epsilon}{dx} \right) \right) w dx = 0, \\ \langle \frac{\partial L}{\partial \theta}, t \rangle &= \int_0^1 \theta(x) t(x) dx + \int_0^1 v_\epsilon(x) t(x) dx \\ &= \int_0^1 (\theta(x) + v_\epsilon(x)) t(x) dx = 0, \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} \beta(u_\epsilon - U) + \frac{d}{dx} \left( a_\epsilon(x) \frac{dv_\epsilon}{dx} \right) &= 0, \\ \theta(x) + v_\epsilon(x) &= 0. \end{aligned}$$

i.e.

$$-\frac{d}{dx} \left( a_\epsilon(x) \frac{dv_\epsilon}{dx} \right) - \beta u_\epsilon = -\beta U(x),$$

$$v_\epsilon(x) = -\theta(x).$$

Hence,  $v(x) = -\theta(x)$  and the optimal problem is equivalent to the following partial differential equation problem,

$$\begin{cases} -\frac{d}{dx} \left( a_\epsilon(x) \frac{du_\epsilon}{dx} \right) + v_\epsilon(x) = f(x) & x \in (0, 1), \\ -\frac{d}{dx} \left( a_\epsilon(x) \frac{dv_\epsilon}{dx} \right) - \beta u_\epsilon(x) = -\beta U(x) & x \in (0, 1) \\ u_\epsilon = 0 & x = 0, 1, \\ v_\epsilon = 0 & x = 0, 1. \end{cases} \quad (2.3.1)$$

## 2.4 Numerical Example

### 2.4.1 Discretization of the system

Now let's look several numerical examples. Finite difference method will be applied here to find the numerical solution. The discretization of (2.3.1) is given as below

$$-\frac{a_{\epsilon, i+\frac{1}{2}} u_{i+1} - (a_{\epsilon, i+\frac{1}{2}} + a_{\epsilon, i-\frac{1}{2}}) u_i + a_{\epsilon, i-\frac{1}{2}} u_{i-1}}{h^2} + v_i = f_i, i = 1, 2, \dots, n-1 \quad (2.4.1)$$

$$-\frac{a_{\epsilon, i+\frac{1}{2}} v_{i+1} - (a_{\epsilon, i+\frac{1}{2}} + a_{\epsilon, i-\frac{1}{2}}) v_i + a_{\epsilon, i-\frac{1}{2}} v_{i-1}}{h^2} - \beta u_i = -\beta U_i, i = 1, 2, \dots, n-1 \quad (2.4.2)$$

$$u_0 = u_n = 0 \quad (2.4.3)$$

$$v_0 = v_n = 0, \quad (2.4.4)$$

where for any function  $g(x)$ ,  $g_i = g(x_i)$  and  $0 = x_0 < x_1 < \dots < x_n = 1$  is a uniform grid, with grid spacing  $\Delta x = h = 1/n$ . We will choose  $a_\epsilon$  from an example in [2]. Let  $a_\epsilon = \frac{1}{2 + 1.8 \sin(\frac{2\pi x}{\epsilon})}$ ,  $\beta = 100,000$  and  $U(x) = \sin(2\pi x)$ ,  $f(x) = x^2$ . First, let's look at the graph of coefficient  $a_\epsilon(x)$  with different values of  $\epsilon$ . Figure 2.1 shows the oscillation of the coefficient with small  $\epsilon$  and the smaller the  $\epsilon$  is, the more oscillated the coefficient is. Next, we will look at several examples for different values of  $\epsilon$ .

- (i)  $\epsilon = 0.01$ ,  $\Delta x = \frac{1}{2000}$ , the graphs of  $u$  and  $v$  are shown in Figure 2.2 and Figure 2.3.

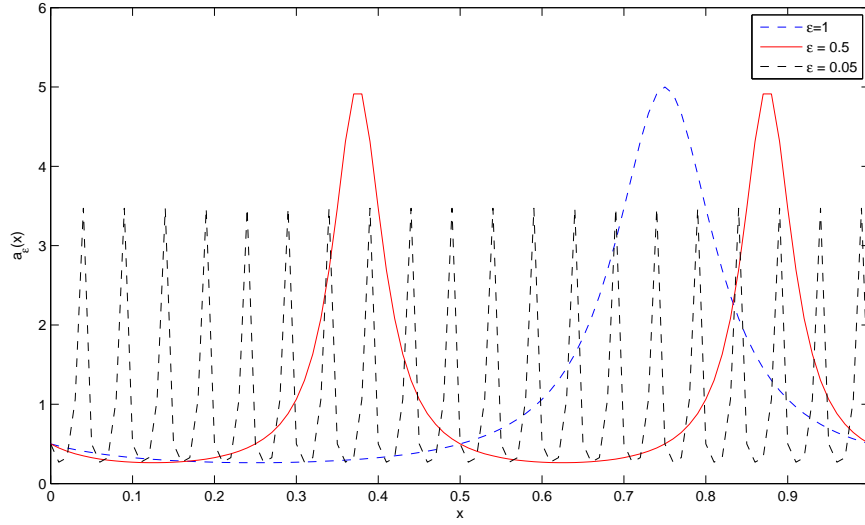


Figure 2.1 The graph of  $a_\epsilon(x)$  with  $\epsilon = 1, \epsilon = 0.5$  and  $\epsilon = 0.05$ .

(ii)  $\epsilon = 0.001, \Delta x = \frac{1}{2000}$ , the graphs of  $u$  and  $v$  are shown in Figure 2.4 and Figure 2.5.

From Figure 2.2 and Figure 2.4, we could find that, the  $u_\epsilon$  approximates the target function  $U(x) = \sin(2\pi x)$  very well, when  $\epsilon$  is small enough. This special case was studied by Kesavan and Vanninathan, who assumed that  $a_\epsilon$  is periodic. For the following problem,

$$\begin{cases} -\frac{d}{dx}\left(a_0 \frac{du_0}{dx}\right) + v_0(x) = f(x) & x \in (0, 1), \\ -\frac{d}{dx}\left(a_0 \frac{dv_0}{dx}\right) - \beta u_0(x) = -\beta U(x) & x \in (0, 1) \\ u_0 = v_0 = 0 & x = 0, 1, \end{cases} \quad (2.4.5)$$

where  $a_0$  is a constant and they proved that  $a_0$  was indeed the limit of  $a_\epsilon$  in the topology of H-convergence. Also for the periodic  $a_\epsilon$  of the one-dimensional case, they also gave its limit of H-convergence, which is

$$a_0 = \left[ m \left( \frac{1}{a} \right) \right]^{-1},$$

where  $m(h) = \int_0^1 h(y) dy$  for a periodic function  $h$  on  $[0, 1]$ .

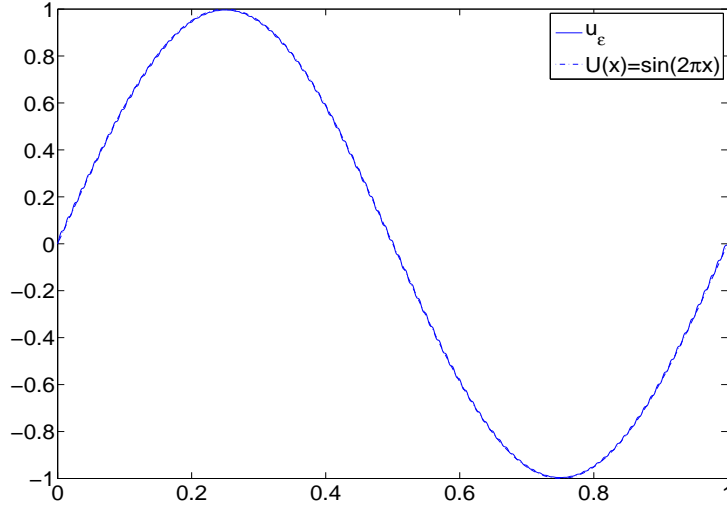


Figure 2.2 The graph of  $u_\epsilon$  and  $U(x) = \sin(2\pi x)$ ,  $0 \leq x \leq 1$ ,  $\epsilon = 0.01$

#### 2.4.2 Comparison of the results for $a_\epsilon$ and $a_0$

We will compare the relationship between

$$\begin{cases} -\frac{d}{dx}(a_\epsilon(x) \frac{du_\epsilon}{dx}) + v_\epsilon(x) = f(x) & x \in (0, 1), \\ -\frac{d}{dx}(a_\epsilon(x) \frac{dv_\epsilon}{dx}) - \beta u_\epsilon(x) = -\beta U(x) & x \in (0, 1) \\ u_\epsilon = v_\epsilon = 0 & x = 0, 1, \end{cases} \quad (2.4.6)$$

and

$$\begin{cases} -\frac{d}{dx}(a_0 \frac{du_0}{dx}) + v_0(x) = f(x) & x \in (0, 1), \\ -\frac{d}{dx}(a_0 \frac{dv_0}{dx}) - \beta u_0(x) = -\beta U(x) & x \in (0, 1) \\ u_0 = v_0 = 0 & x = 0, 1, \end{cases} \quad (2.4.7)$$

with two numerical examples. Like the prior example, let's suppose  $a_\epsilon = \frac{1}{2 + 1.8 \sin(\frac{2\pi x}{\epsilon})}$ ,  $\beta = 100,000$  and  $U(x) = \sin(2\pi x)$ ,  $f(x) = x^2$ . Let  $u_\epsilon, v_\epsilon$  denote the numerical solutions of effective equations with  $a_\epsilon$  and  $u_0, v_0$  denote the numerical solutions of partial differential equations with coefficient  $a_0$ . Hence,

$$a_0 = \left[ m \left( \frac{1}{a} \right) \right]^{-1} = \left[ \int_0^1 (2 + 1.8 \sin(2\pi y)) dy \right]^{-1} = \frac{1}{2}.$$

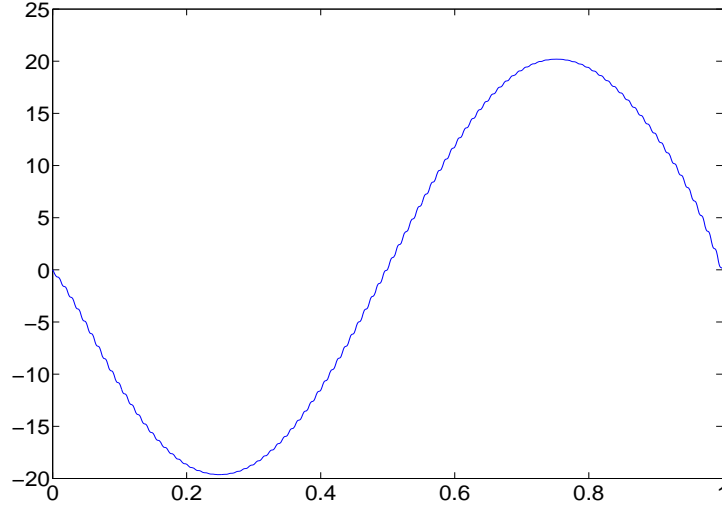


Figure 2.3 The graph of  $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.01$

We will give several graphs to illustrate the errors between  $u_\epsilon$  and  $u_0$ ,  $v_\epsilon$  and  $v_0$  for different values of  $\epsilon$ .

- (i)  $\epsilon = 0.01, \Delta x = \frac{1}{2000}$ , the graphs of  $u_\epsilon, v_\epsilon, u_0, v_0$  and their errors are shown in Figure 2.6, Figure 2.7, Figure 2.8, Figure 2.9.
- (ii)  $\epsilon = 0.001, \Delta x = \frac{1}{2000}$ , the graphs of  $u_\epsilon, v_\epsilon, u_0, v_0$  and their errors are shown in Figure 2.10, Figure 2.11, Figure 2.12, Figure 2.13.

From Figure 2.7 and Figure 2.9, we can find the oscillation of the error of  $u_\epsilon$  and  $u_0$ . Therefore, we can find a test function, such that  $u_\epsilon$  is weak convergent to  $u_0$ . Analogously, the error of  $v_\epsilon$  and  $v_0$  also has such oscillation, which means that  $v_\epsilon$  is also weak convergent to  $v_0$ .

### 2.4.3 Comparison of values of objective functions

We are going to compare the value of the objective function that is from the original optimal control system and the one from the effective system. Let's take  $\epsilon = 0.001, \Delta x = \frac{1}{2000}$  for example. With numerical solution of (2.4.6) and (2.4.7).

For (2.4.6), the objective function is defined as

$$J(\theta) = \frac{\beta}{2} \int_0^1 |u_\epsilon - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx, \quad (2.4.8)$$

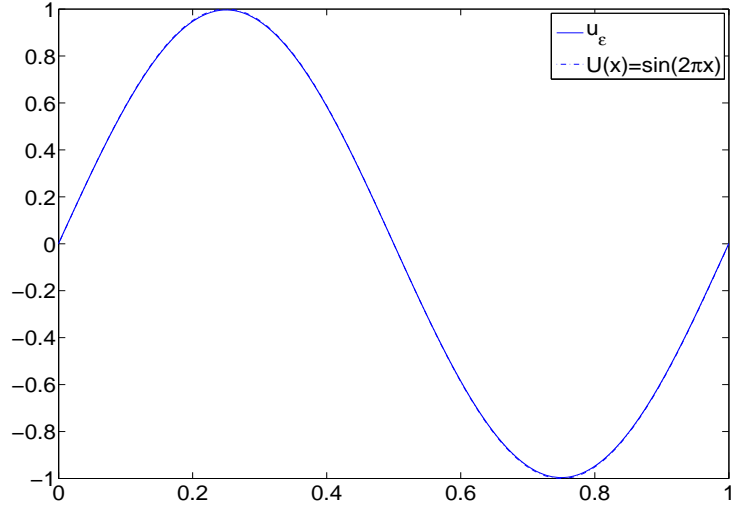


Figure 2.4 The graph of  $u_\epsilon$  and  $U(x) = \sin(2\pi x)$ ,  $0 \leq x \leq 1$ ,  $\epsilon = 0.001$

Table 2.1 Comparison of values of objective functions

	Original system (2.4.6)	Effective system (2.4.7)
Values of J	100.3122	100.2463

where  $\theta(x) = -v(x)$ .

For (2.4.7), the objective function is defined as

$$J(\theta) = \frac{\beta}{2} \int_0^1 |u_0 - U|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx, \quad (2.4.9)$$

where  $\theta(x) = -v_0(x)$ . Table 2.1 shows the values of the objective functions from these two different systems, and we find that the value of objective function from effective system is a good approximation for the one from original system.



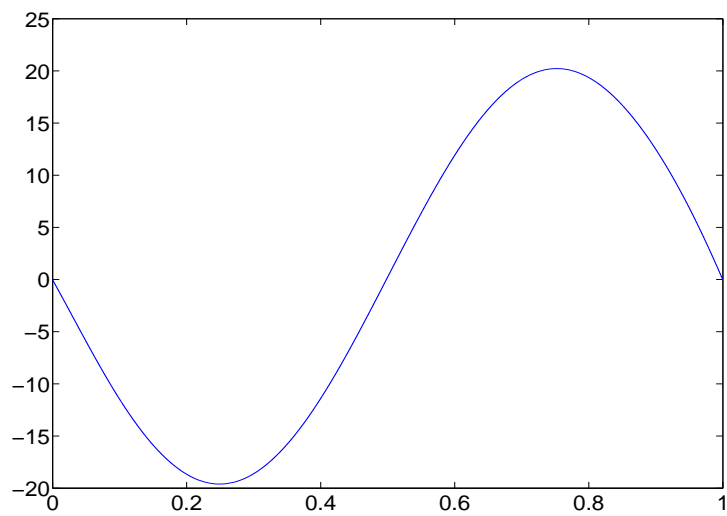


Figure 2.5 The graph of  $v_\epsilon, 0 \leq x \leq 1, \epsilon = 0.001$

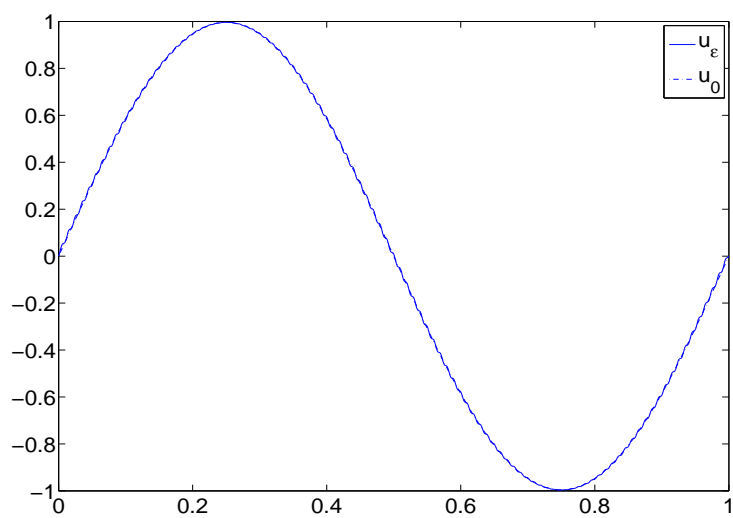


Figure 2.6 The graph of  $u_\epsilon$  and  $u_0, \epsilon = 0.01$

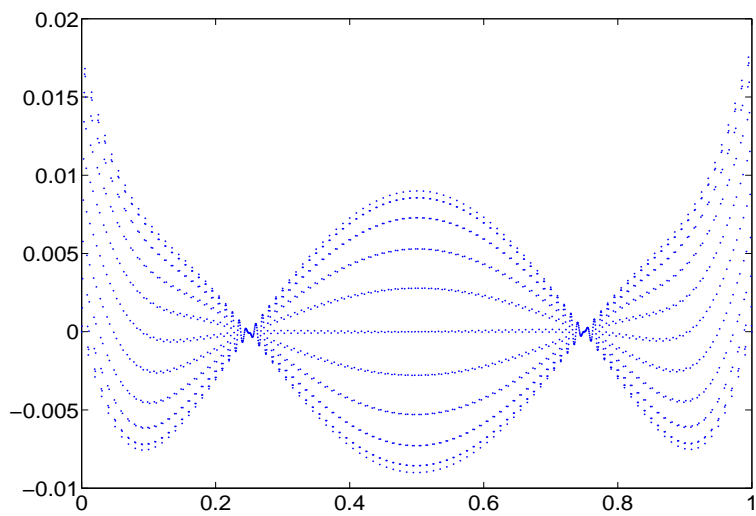


Figure 2.7 The graph of error between  $u_\epsilon$  and  $u_0$ ,  $\epsilon = 0.01$

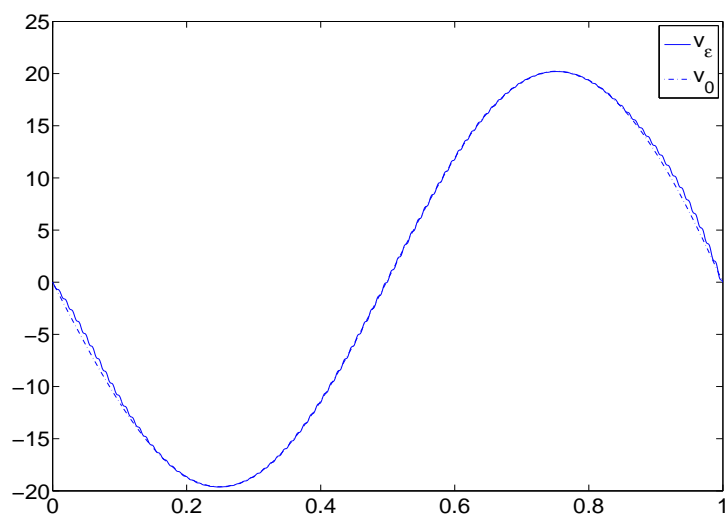


Figure 2.8 The graph of  $v_\epsilon$  and  $v_0$ ,  $\epsilon = 0.01$

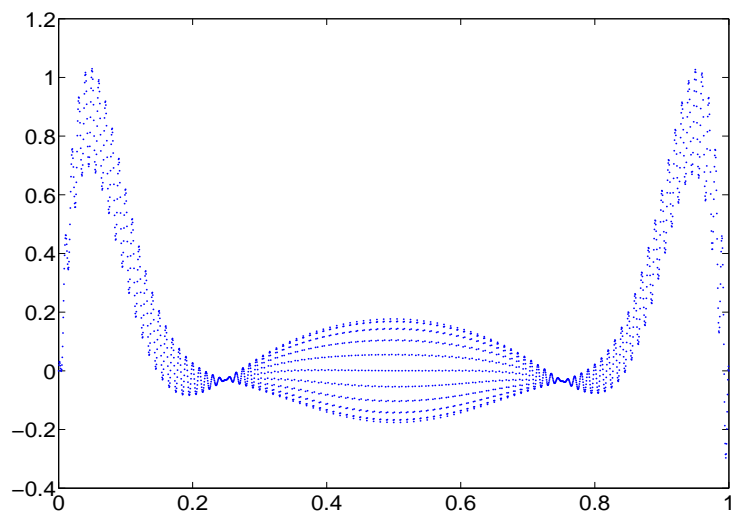


Figure 2.9 The graph of error between  $v_\epsilon$  and  $v_0$ ,  $\epsilon = 0.01$

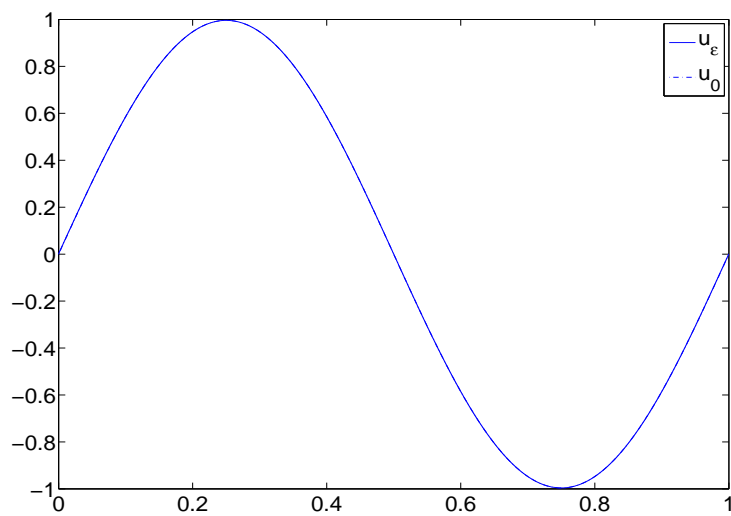


Figure 2.10 The graph of  $u_\epsilon$  and  $u_0$ ,  $\epsilon = 0.001$

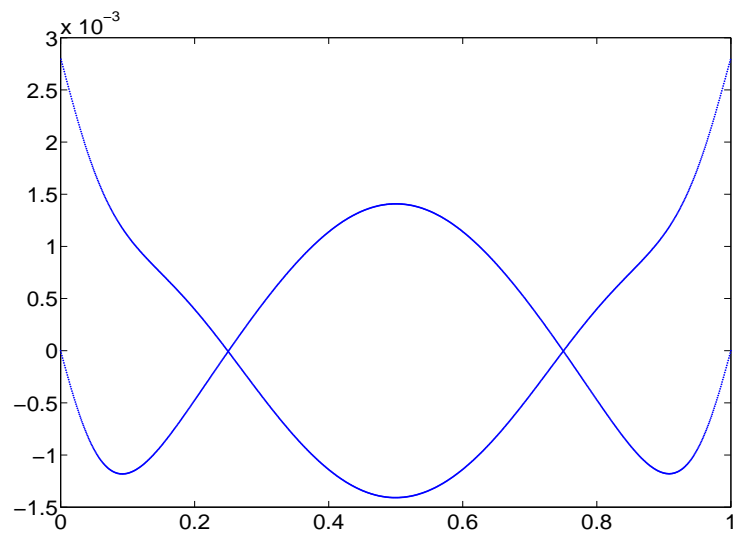


Figure 2.11 The graph of error between  $u_\epsilon$  and  $u_0$ ,  $\epsilon = 0.001$

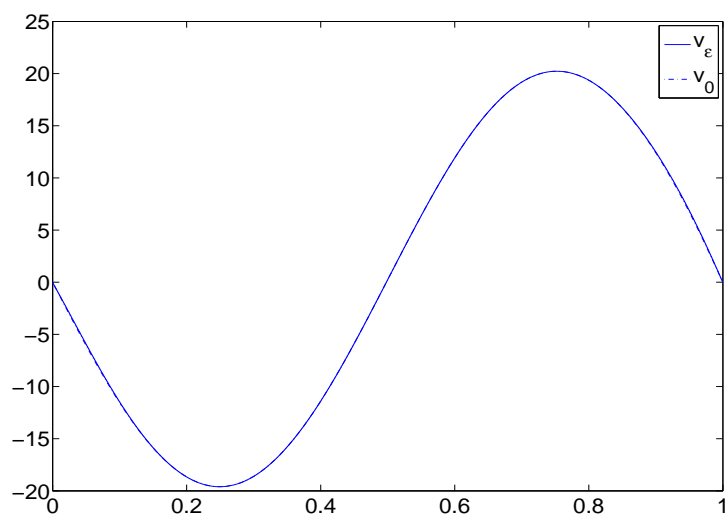


Figure 2.12 The graph of  $v_\epsilon$  and  $v_0$ ,  $\epsilon = 0.001$

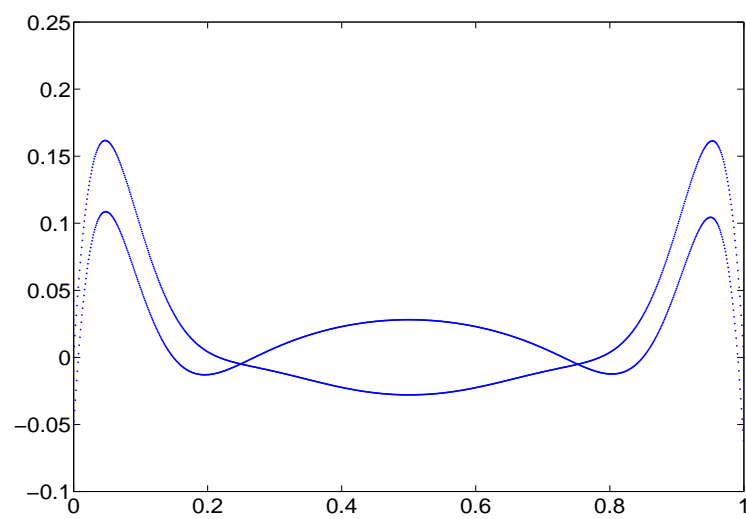


Figure 2.13 The graph of error between  $v_\epsilon$  and  $v_0$ ,  $\epsilon = 0.001$

### CHAPTER 3. Stochastic Homogenization for Elliptic Equations with Ergodic assumption

#### 3.1 Introduction

Let  $(\Omega, \mu)$  be a probability space, and

$$a^T(x, \omega) = a(T(x)\omega),$$

where  $\omega$  is a random variable and  $T = T(x) : \Omega \rightarrow \Omega$  is a transformation which preserves the measure  $\mu$  on  $\Omega$ .  $T(x) : \Omega \rightarrow \Omega$ ,  $x \in \mathbb{R}^m$  is called a dynamical system with  $m$ -dimensional time, or simply an  $m$ -dimensional dynamical system if it satisfies the following conditions:

- (i)  $T(0) = I$  ( $I$  is the identity mapping), and  $T(x + y) = T(x)T(y)$ ,  $\forall x, y \in \mathbb{R}^m$ .
- (ii)  $T(x) : \Omega \rightarrow \Omega$  preserves the measure  $\mu$  on  $\Omega$ , i.e. for every  $x \in \mathbb{R}^m$  and every  $u$  measurable set  $\mathfrak{F} \in \Omega$ , we have

$$T(x)\mathfrak{F} \text{ is measurable} \quad \mu(T(x)\mathfrak{F}) = \mu(\mathfrak{F}).$$

- (iii) For any measurable function  $f(\omega)$  on  $\Omega$ , the function  $f(T(x)\omega)$  defined on the  $\Omega \times \mathbb{R}^m$  is also measurable.

Suppose  $a : \Omega \rightarrow \mathbb{R}$  satisfies the following,

- (1)  $a(\cdot)$  is  $\mu$ -measurable.
- (2) There exists positive constants  $\alpha$  and  $\beta$  such that

$$\alpha \leq a(\omega) \leq \beta \quad \text{a.e. in } \Omega.$$

Let  $a_\epsilon^T(x, \omega) = a^T\left(\frac{x}{\epsilon}, \omega\right)$ , and we consider the following problem,

$$\begin{cases} -\left(a_\epsilon^T(x, \omega)u'_\epsilon(x, \omega)\right)' = f(x), x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0. \end{cases}$$

Let

$$\bar{a} = \frac{1}{E\left\{\frac{1}{a}\right\}}$$

where  $E\left\{\frac{1}{a}\right\}$  denotes the expression

$$E\left\{\frac{1}{a}\right\} = \int_{\Omega} \frac{1}{a(\omega)} d\mu(\omega)$$

and we will show that the original problem will admit a homogenization problem with above deterministic coefficient.

### 3.2 Birkhoff Ergodic Theorem

We will also take one dimensional case for example.

**Definition:**(Mean Value [20])

Let  $f \in L^1_{loc}(\mathbb{R})$  and suppose the limit  $\lim_{\epsilon \rightarrow 0} \frac{1}{|K|} \int_K f\left(\frac{x}{\epsilon}\right) dx$  exists for any Lebesgue measurable subset  $K \subset \mathbb{R}$  independently of  $K$ . Then, we say that  $f$  has a mean value  $M\{f\}$  given by

$$M\{f\} = \lim_{\epsilon \rightarrow 0} \frac{1}{|K|} \int_K f\left(\frac{x}{\epsilon}\right) dx.$$

**Theorem:**(Birkhoff Theorem [7])

Let  $f \in L^p(\Omega)$ ,  $p \geq 1$ . Then for almost all  $\omega \in \Omega$  the realization  $f(T(x)\omega)$  possesses a mean value in the following sense,

$$f\left(\frac{x}{\epsilon}\right) \rightharpoonup M\{f\} \text{ in } L^p_{loc}(\mathbb{R}).$$

Moreover, the mean value  $M\{f(T(x)\omega)\}$ , considered as a function of  $\omega \in \Omega$ , is invariant, and

$$\langle f \rangle \equiv \int_{\Omega} f(\omega) d\mu = \int_{\Omega} M\{f(T(x)\omega)\} d\mu.$$

In particular, if the system  $T(x)$  is ergodic, then

$$M\{f(T(x)\omega)\} = \langle f \rangle \quad \text{for almost all } \omega \in \Omega.$$

Let's first consider the following linear elliptic differential equation with Dirichlet boundary conditions

$$\begin{cases} -\frac{d}{dx} \left( a_\epsilon(x) \frac{d}{dx} u_\epsilon(x) \right) = f(x), x \in L = (0, 1), \\ u_\epsilon(0) = u_\epsilon(1) = 0. \end{cases} \quad (3.2.1)$$

where  $f \in L^2(\Omega)$  and  $a_\epsilon = a\left(\frac{x}{\epsilon}\right)$ ,  $a(x)$  is periodic and satisfies  $\alpha \leq a(x) \leq \beta$  where  $\alpha, \beta > 0$ .

As for the data, we assume that

$$f \in H^{-1}((0, 1); \mathbb{R}) = (H_0^1(0, 1); \mathbb{R})^*.$$

Under natural suppositions on the coefficient function  $a$  and the data  $f$  given above, the boundary problem given above does possess a unique solution  $u_\epsilon \in H_0^1((0, 1); \mathbb{R})$ .

**Theorem:** Let  $u_\epsilon(x)$  be solution of above differential equation with Dirichlet boundary conditions. Then one has the convergence relation  $u_\epsilon \rightarrow \bar{u}$  in  $L^2(0, 1)$ , where  $\bar{u}$  satisfies

$$\begin{cases} -\frac{d}{dx} \left( \bar{a}(x) \frac{d}{dx} \bar{u}(x) \right) = f(x), x \in L = (0, 1), \\ \bar{u}(0) = \bar{u}(1) = 0. \end{cases} \quad (3.2.2)$$

where  $\bar{a} = \frac{1}{\int \frac{1}{a(x)} dx}$ .

**Proof:** For any  $\phi \in H_0^1(0, 1)$  with the first equation of (3.2.2), we have

$$-\int \frac{d}{dx} \left( a_\epsilon(x) \frac{d}{dx} u_\epsilon(x) \right) \phi dx = \int f \phi dx.$$

By integration by parts,

$$\int a_\epsilon(x) u'_\epsilon(x) \phi' dx = \int f \phi dx.$$

Let  $\phi = u_\epsilon$  in above equation with assumptions for  $a(s)$ ,

$$\begin{aligned} \alpha \|u'_\epsilon\|_{L^2}^2 &\leq \int a_\epsilon(x) u'_\epsilon(x) u'_\epsilon(x) dx = \int f u'_\epsilon(x) dx \\ &\leq \|f\|_{L^2} \|u_\epsilon\|_{L^2} \\ &\leq C_1 \|f\|_{L^2} \|u'_\epsilon\|_{L^2}, \end{aligned}$$

where we have used Poincaré inequality in the last step. Then we have

$$\|u_\epsilon\|_{L^2} \leq \frac{C_1}{\alpha} \|f\|_{L^2}.$$



Therefore,

$$\|u'_\epsilon\|_{L^2} \leq C_1 \|u_\epsilon\|_{L^2} \leq \frac{C_2}{\alpha} \|f\|_{L^2}.$$

Hence,  $u_\epsilon$  is bounded in  $H_0^1(0, 1)$ , and then there exists  $\bar{u} \in H_0^1(0, 1)$ , such that

$$u_\epsilon \rightharpoonup \bar{u} \text{ in } H_0^1(0, 1).$$

Since  $H_0^1(0, 1)$  is compactly imbedded in  $L^2(0, 1)$ , by Rellich's Theorem,

$$u_\epsilon \rightarrow \bar{u} \text{ in } L^2(0, 1).$$

There also exists  $\sigma$  such that  $a_\epsilon(x)u'_\epsilon(x) \rightharpoonup \sigma$  in  $H_0^1(0, 1)$  and  $a_\epsilon(x)u'_\epsilon(x) \rightarrow \sigma$  in  $H_0^1(0, 1)$ . Since  $u'_\epsilon(x) = (a_\epsilon(x)u'_\epsilon(x)) \frac{1}{a_\epsilon(x)}$  and  $\frac{1}{a_\epsilon(x)} \rightharpoonup \int \frac{1}{a(x)} dx$  in  $L^2(0, 1)$ ,

$$(a_\epsilon(x)u'_\epsilon(x)) \frac{1}{a_\epsilon(x)} \rightharpoonup \sigma \int \frac{1}{a(x)} dx.$$

Since  $u'_\epsilon \rightharpoonup \bar{u}'$  in  $L^2(0, 1)$ ,

$$\bar{u}' = \sigma \int \frac{1}{a(x)} dx.$$

Let  $\bar{a} = \frac{1}{\int \frac{1}{a(x)} dx}$ . Then one has

$$\sigma = \bar{a}\bar{u}' \text{ and } -\sigma' = f.$$

Hence,

$$-(\bar{a}\bar{u}')' = f.$$

Then let  $\omega \in \Omega$  be fixed but arbitrary and let's go back the original problem with random coefficient,

$$\begin{cases} -(a_\epsilon^T(x, \omega)u'_\epsilon(x, \omega))' = f(x), x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0. \end{cases}$$

From the proof of previous theorem, we know that  $\{u_\epsilon(\cdot, \omega)\}_{\epsilon>0}$  is bounded in  $H_0^1(0, 1)$  and then

$$u_\epsilon(\cdot, \omega) \rightharpoonup \bar{u}(\omega) \text{ in } H_0^1(0, 1),$$

and so

$$u'_\epsilon \rightarrow \bar{u}'(\omega) \text{ in } L^2(0, 1). \quad (3.2.3)$$

Similarly, let  $\sigma_\epsilon(x, \omega) = a_\epsilon^T(x, \omega)u'_\epsilon(x, \omega)$  and then

$$\sigma_\epsilon(\cdot, \omega) \rightharpoonup \bar{\sigma}(\cdot, \omega) \quad \text{in } H^1(0, 1).$$

By compact imbedding of  $H^1(0, 1)$  into  $L^2(0, 1)$ , one can get that

$$\sigma_\epsilon(\cdot, \omega) \rightarrow \bar{\sigma}(\cdot, \omega) \quad \text{in } L^2(0, 1).$$

Therefore,

$$u'_\epsilon(\cdot, \omega) = \frac{\sigma_\epsilon(\cdot, \omega)}{a_\epsilon^T(\cdot, \omega)} \rightharpoonup \frac{\bar{\sigma}(\cdot, \omega)}{\bar{a}} \quad \text{in } L^2(0, 1), \quad (3.2.4)$$

where  $\bar{a} = \frac{1}{E\{\frac{1}{a}\}}$ . By (0.1) and (0.2), one can get that

$$\frac{\bar{\sigma}(\cdot, \omega)}{\bar{a}} = \bar{u}'(\cdot, \omega), \quad \text{i.e. } \bar{\sigma}(\cdot, \omega) = \bar{a}\bar{u}'(\cdot, \omega).$$

Moreover, we know that  $-\sigma'_\epsilon(\cdot, \omega) = f$  for all  $\epsilon$  and therefore

$$-\bar{\sigma}'(\cdot, \omega) = f$$

and so

$$\begin{cases} -(\bar{a}\bar{u}'(x, \omega))' = f(x), x \in L = (0, 1), \\ \bar{u}(0, \omega) = \bar{u}(1, \omega) = 0, \end{cases}$$

where  $\omega$  is fixed but arbitrary. By the theory of elliptic partial differential equations, one can get that the solution of above elliptic problem exists and unique, which means that the solution  $\bar{u}$  is independent of  $\omega$ . Hence, we have

$$\begin{cases} -(\bar{a}\bar{u}'(x))' = f(x), x \in L = (0, 1), \\ \bar{u}(0) = \bar{u}(1) = 0. \end{cases}$$

Let's consider the optimal control problem

$$\begin{cases} -(a_\epsilon^T(x, \omega)u'_\epsilon(x, \omega))' = f(x) + \theta(x), x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0, \end{cases} \quad (3.2.5)$$

and define the objective function

$$J(\theta) = \frac{\beta}{2}E \left[ \int_0^1 |u_\epsilon - U|^2 dx \right] + \frac{1}{2} \int_0^1 \theta^2(x) dx, \quad (3.2.6)$$

where the target function  $U(x)$  is deterministic and  $f(x)$  is a continuous function. Note that there is a unique weak solution for (3.2.5) by Lax-Milgram theorem and the error estimate for the solutions will be similar with what we talked before. Both will be used for the error estimates for the optimality system of equations. The optimal control problem (3.2.5) will become

$$\begin{cases} -(\bar{a}\bar{u}'(x))' = f(x) + \theta(x), x \in L = (0, 1), \\ \bar{u}(0) = \bar{u}(1) = 0, \end{cases} \quad (3.2.7)$$

with objective function

$$J(\theta) = \frac{\beta}{2} \int_0^1 |\bar{u}(x) - U(x)|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx. \quad (3.2.8)$$

### 3.3 Numerical Example

In this section, we are going to consider a numerical example for the stochastic optimization problem with ergodic coefficient. First, let's look at an example for  $a_\epsilon^T(x, \omega)$  with ergodicity property [5],

$$T(\omega) = \begin{cases} 0 & \text{if } \omega = 0, \\ \frac{1}{\omega} \bmod 1 & \text{if } 0 < \omega < 1. \end{cases} \quad (3.3.1)$$

One can easily show that the continued fraction map does not preserve Lebesgue measure, i.e. there exists  $B \in \mathcal{B}$  such that  $T^{-1}B$  and  $B$  have different measure. (Indeed choose  $B$  to be any interval.) Although the continued fraction map does not preserve Lebesgue measure, it does preserve Gauss measure  $\mu$ , defined by

$$\mu(B) = \frac{1}{\ln 2} \int_B \frac{1}{1+s} ds. \quad (3.3.2)$$

The measure  $\mu$  defined above is invariant. Here is a simple proof in [7].

$$\begin{aligned}
\mu(T^{-1}(0, \alpha)) &= \mu\left(\bigcup_{k=1}^{\infty} \left(\frac{1}{k+\alpha}, \frac{1}{k}\right)\right) = \sum_{k=1}^{\infty} \mu\left(\left(\frac{1}{k+\alpha}, \frac{1}{k}\right)\right) \\
&= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int_{1/(k+\alpha)}^{1/k} \frac{dx}{1+x} \\
&= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left[ \log\left(1 + \frac{1}{k}\right) - \log\left(1 + \frac{1}{k+\alpha}\right) \right] \\
&= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left[ \log\left(1 + \frac{\alpha}{k}\right) - \log\left(1 + \frac{\alpha}{k+\alpha}\right) \right] \\
&= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int_{\alpha/(k+1)}^{\alpha/k} \frac{dx}{1+x} \\
&= \frac{1}{\ln 2} \int_0^{\alpha} \frac{dx}{1+x} = \mu((0, \alpha)).
\end{aligned}$$

Therefore, by theorems in [7] we have  $T$  with invariant measure  $\mu$  is ergodic.

Then let  $a(s) = 1 + s^2$ ,  $s \in [0, 1]$  and  $1 \leq a \leq 2$ . Let  $T$  be continued fraction as defined above and

$$T_x(\omega) = x \cdot T(\omega)$$

and therefore,

$$a_{\epsilon}^T(x, \omega) = a(T_{x/\epsilon}(\omega)) = a\left(\frac{x}{\epsilon} \cdot T(\omega)\right) = 1 + \left(\frac{x}{\epsilon}\right)^2 (T(\omega))^2. \quad (3.3.3)$$

By the theorem that we have claimed before, we obtain that

$$\frac{1}{a_{\epsilon}^T(x, \omega)} \rightharpoonup E\left\{\frac{1}{a}\right\} \quad \text{in } L^2(0, 1)$$

For  $E\left\{\frac{1}{a}\right\}$ , we have

$$E\left\{\frac{1}{a}\right\} = \int_{\Omega} \frac{1}{a(s)} d\mu = \frac{1}{\ln 2} \int_0^1 \frac{1}{(1+s)a(s)} ds.$$

For the  $T$  and  $a(\cdot)$  defined above, we obtain that

$$\begin{aligned}
E\left\{\frac{1}{a}\right\} &= \frac{1}{\ln 2} \int_0^1 \frac{1}{(1+s)a(s)} ds \\
&= \frac{1}{\ln 2} \int_0^1 \frac{1}{(1+s)(1+s^2)} ds \\
&= 0.816545,
\end{aligned} \quad (3.3.4)$$

and

$$\bar{a} = \frac{1}{E \left\{ \frac{1}{a} \right\}} = 1.2247. \quad (3.3.5)$$

Consider the following homogenized optimal control problem

$$\begin{cases} -(\bar{a}\bar{u}'(x))' = f(x) + \theta(x), x \in L = (0, 1), \\ \bar{u}(0) = \bar{u}(1) = 0, \end{cases} \quad (3.3.6)$$

with objective function

$$J(\theta) = \frac{\beta}{2} \int_0^1 |\bar{u}(x) - U(x)|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx. \quad (3.3.7)$$

Based on the analysis in previous chapter, the effective equations of (3.3.6) and (3.3.7) are

$$\begin{cases} -\frac{d}{dx}(\bar{a}\frac{d\bar{u}}{dx}) + v(x) = f(x) & x \in (0, 1), \\ -\frac{d}{dx}(\bar{a}\frac{dv}{dx}) - \beta\bar{u}(x) = -\beta U(x) & x \in (0, 1) \\ \bar{u}(x) = v(x) = 0 & x = 0, 1, \end{cases} \quad (3.3.8)$$

where  $v(x) = -\theta(x)$ ,  $f(x) = x^2$  and  $U(x) = \sin(2\pi x)$ . Make a partition of  $[0, 1]$  into  $n$  subintervals and the length of each subinterval is  $\Delta x = 1/n$  and  $x_i = i\Delta x$ ,  $i = 0, 1, \dots, n$ .

Let's consider (3.3.8) at  $x_i, i = 1, 2, \dots, n-1$  and the finite difference discretization is

$$-\frac{\bar{a}_{i+\frac{1}{2}}u_{i+1} - (\bar{a}_{i+\frac{1}{2}} + \bar{a}_{i-\frac{1}{2}})\bar{u}_i + \bar{a}_{i-\frac{1}{2}}\bar{u}_{i-1}}{h^2} + v_i = f_i, i = 1, 2, \dots, n-1 \quad (3.3.9)$$

$$-\frac{\bar{a}_{i+\frac{1}{2}}v_{i+1} - (\bar{a}_{i+\frac{1}{2}} + \bar{a}_{i-\frac{1}{2}})v_i + \bar{a}_{i-\frac{1}{2}}v_{i-1}}{h^2} - \beta\bar{u}_i = -\beta U_i, i = 1, 2, \dots, n-1 \quad (3.3.10)$$

$$\bar{u}_0 = \bar{u}_n = 0 \quad (3.3.11)$$

$$v_0 = v_n = 0. \quad (3.3.12)$$

Figure 3.1, Figure 3.2 and Figure 3.3 show the oscillation of coefficient  $a_\epsilon^T(x, \omega)$  for different values of  $\epsilon$ .

Figure 3.4 shows the  $\bar{u}(x)$  and target function  $U(x)$ .

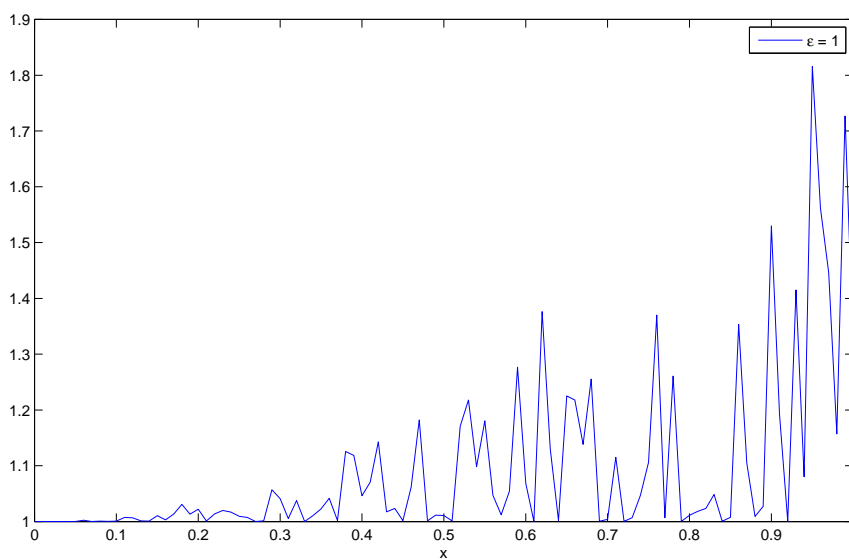


Figure 3.1 The graph of  $a_{\epsilon}^T(x, \omega)$  with  $\epsilon = 1$ .

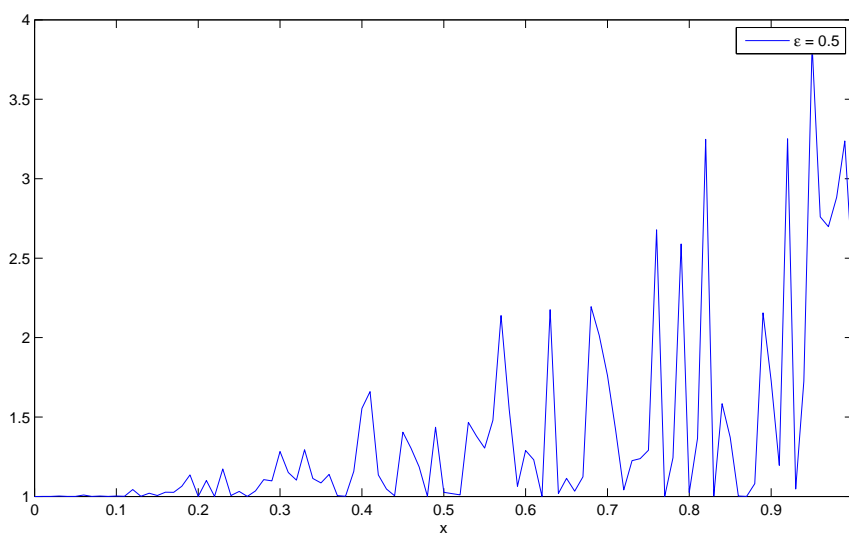


Figure 3.2 The graph of  $a_{\epsilon}^T(x, \omega)$  with  $\epsilon = 0.5$ .

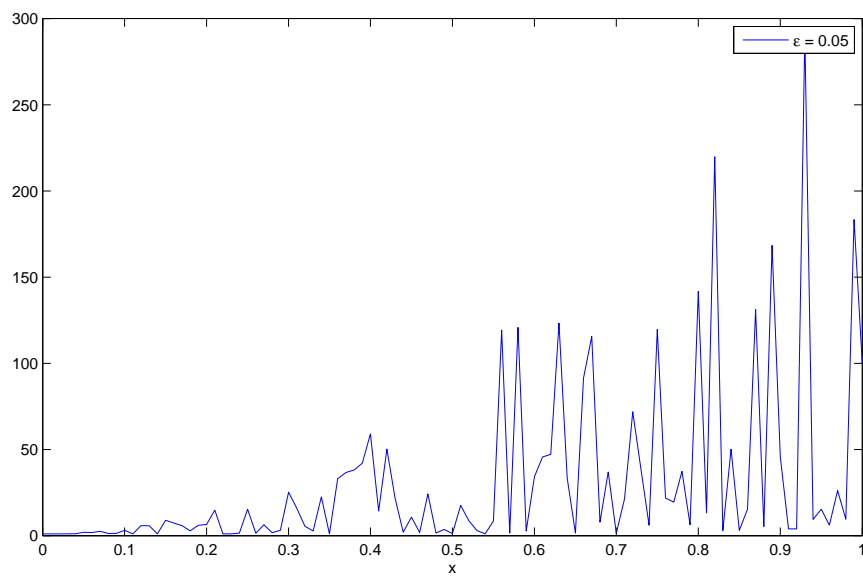


Figure 3.3 The graph of  $a_\epsilon^T(x, \omega)$  with  $\epsilon = 0.05$ .

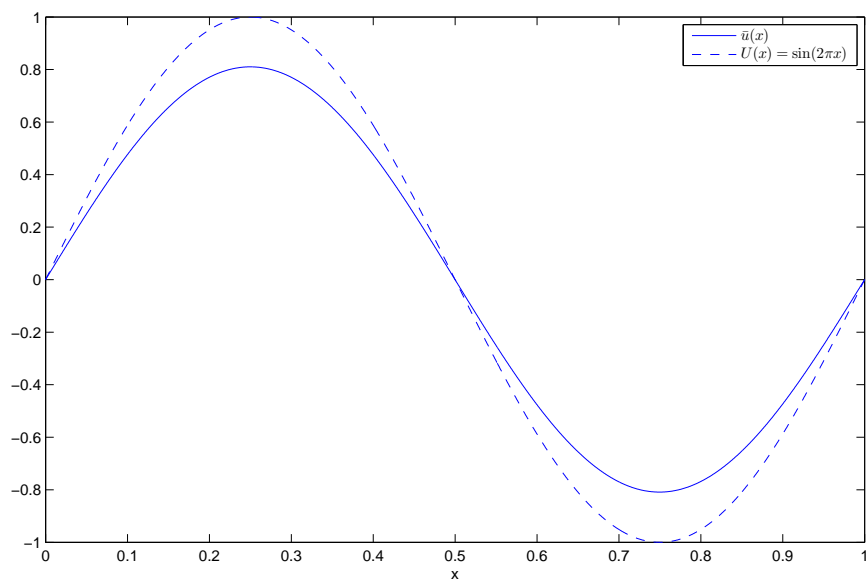


Figure 3.4 The graph of  $\bar{u}(x)$  and target function  $U(x) = \sin(2\pi x)$

## CHAPTER 4. Wiener Chaos Expansion for Stochastic Elliptic Equation and Optimal Control

### 4.1 Wiener Chaos Expansion and Hermite Polynomials

In this section, we will develop the general framework to be used in the following part regarding Wiener chaos expansion. We will start with some basic definitions and properties of white noise probability space. Then Bochner-Minlos theorem, the Wiener-Itô chaos expansion, Kondratiev spaces and wick products will be also discussed in this section.

#### 4.1.1 Brocher-Minlos Theorem

Let  $S(\mathbb{R}^d)$  be the Schwartz space of rapidly decreasing smooth ( $C^\infty$ ) real-valued function on  $\mathbb{R}^d$ . Under the family of seminorms

$$\|f\|_{k,\alpha} = \sup_{x \in \mathbb{R}^d} \{(1 + |x|^k) |\partial^\alpha f(x)|\}, \quad (4.1.1)$$

$S(\mathbb{R}^d)$  is a Fréchet space, where  $k$  is a non-negative integer,  $\alpha = (\alpha_1, \dots, \alpha_d)$  is a multi-index of non-negative integers  $\alpha_1, \dots, \alpha_d$  and

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} f, \quad \text{where } |\alpha| = \alpha_1 + \dots + \alpha_d. \quad (4.1.2)$$

Let  $S' = S'(\mathbb{R}^d)$  be the dual of  $S(\mathbb{R}^d)$ . We will use this space as our basic probability space and use  $B(S'(\mathbb{R}^d))$  as family of Borel subsets of  $S'(\mathbb{R}^d)$ . The probability measure is given by the following theorem.

**Theorem 4.1.1.** *The Bochner-Minlos theorem [13]* There exists a unique probability measure  $\mu$  on  $B(S'(\mathbb{R}^d))$  with the following property:

$$E[e^{i\langle \cdot, \phi \rangle}] = \int_{S'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad (4.1.3)$$



for all  $\phi \in S(\mathbb{R}^d)$ , where  $\|\phi\|^2 = \|\phi\|_{L^2(\mathbb{R}^d)}^2$ ,  $\langle \omega, \phi \rangle = \omega(\phi)$  is the action of  $\omega \in S'(\mathbb{R}^d)$  on  $E = E_\mu$  denotes the expectation with respect to  $\mu$ .

The proof of this theorem can be found at the appendix of [13]. The triplet  $(S'(\mathbb{R}^d), B(S'(\mathbb{R}^d)), \mu)$  is called the 1-dimensional white noise probability space, and  $\mu$  is called the white noise measure. The following lemma can also be easily obtained.

**Lemma 4.1.1.** Let  $\xi_1, \dots, \xi_n$  be functions in  $S(\mathbb{R}^d)$  that are orthonormal in  $L^2(\mathbb{R}^d)$ . Let  $\lambda_n$  be the normalized Gaussian measure on  $\mathbb{R}^n$ , i.e.,

$$d\lambda_n = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|x|^2} dx_1 \cdots dx_n; x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (4.1.4)$$

Then the random variable

$$\omega \rightarrow (\langle \omega, \xi_1 \rangle, \langle \omega, \xi_2 \rangle, \dots, \langle \omega, \xi_n \rangle) \quad (4.1.5)$$

has distribution  $\lambda_n$ . Equivalently,

$$E[f(\langle \cdot, \xi_1 \rangle, \dots, \langle \cdot, \xi_n \rangle)] = \int_{\mathbb{R}^n} f(x) d\lambda_n \text{ for all } f \in L^1(\lambda_n). \quad (4.1.6)$$

#### 4.1.2 Hermite Polynomials

There are two classical ways of constructing the Wiener-Itô chaos expansion: (a) by Hermite polynomial, (b) by multiple Itô integrals. We are going to discuss the first method in details and the second method can be found in some classical stochastic books.

Consider functions on the real axis  $\mathbf{R} = (-\infty, \infty)$  with Gaussian measure

$$d\mu(x) = \rho(x)dx, \quad \rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \quad (4.1.7)$$

For a probability space  $\Omega$ , the  $L^2$  space with respect to measure  $\mu$  is defined as

$$L^2(\mu) = \left\{ f : \int_{\Omega} f^2(x) d\mu(x) < \infty \right\}. \quad (4.1.8)$$

In the words,

$$L^2(\mu) = \{f : E|f|^2 < \infty\}, \quad (4.1.9)$$

where  $E$  denotes the expectation. The inner product on this space is defined as

$$(f, g)_\mu = \int_{-\infty}^{+\infty} f(x)g(x) d\mu(x) = \int_{-\infty}^{+\infty} f(x)g(x) \rho(x) dx. \quad (4.1.10)$$

We also define

$$(f, g)_\mu = E(f(\zeta)g(\zeta)), \quad (4.1.11)$$

where  $\zeta$  is a standard Gaussian random variable with normal distribution  $N(0, 1)$ . Thus, the Hilbert space  $L^2(\mathbf{R}, \mu)$  can also be interpreted as the space of functions of a unit Gaussian random variable with finite variance.

The Hermite polynomials  $h_n(x)$  are defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \dots \quad (4.1.12)$$

$h_n(x)$  are orthogonal polynomials with respect to the Gaussian measure

$$(h_n, h_m)_\mu = E[h_n(\zeta)h_m(\zeta)] = n!\delta_{nm}.$$

Hence the normalized Hermite polynomials are defined as

$$H_n(x) = \frac{h_n(x)}{\sqrt{n!}} = (n!)^{-\frac{1}{2}} (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \dots \quad (4.1.13)$$

It is well known that  $\{H_n(x) : n = 0, 1, 2, \dots\}$  are a complete orthogonal basis in the Hilbert Space  $L^2(\mathbf{R}, \mu)$  [13]. Since  $H_0(x) = 1$ , particularly we have

$$E[H_n(\zeta)] = \int_{-\infty}^{+\infty} H_n(x) d\mu(x) = (H_n, 1)_\mu = 0, \quad \text{if } n \neq 0. \quad (4.1.14)$$

Therefore, for the Hermite polynomials of order greater than zero, have mean zero.

Differentiating (4.1.12) we get

$$h'_n(x) = xh_n(x) - h_{n+1}(x). \quad (4.1.15)$$

Like most orthogonal polynomials, the generating function of Hermite polynomials is

$$G(x, z) = e^{-\frac{x^2}{2} + xz} = \sum_{n=0}^{\infty} h_n(x) \frac{z^n}{n!} \quad (4.1.16)$$

On the other hand, the Taylor expansion of  $G(x, z)$  with respect to  $z$  is

$$G(x, z) = \sum_{n=0}^{\infty} \frac{\partial^n G(x, z)}{\partial z^n} \Big|_{z=0} \frac{z^n}{n!} \quad (4.1.17)$$

From the definition of  $G(x, z)$ , we have

$$G(x, z) = e^{\frac{x^2}{2}} e^{-\frac{(z-x)^2}{2}} = \sum_{n=0}^{\infty} h_n(x) \frac{z^n}{n!}, \quad (4.1.18)$$

since

$$\left. \frac{\partial e^{-(z-x)^2/2}}{\partial z} \right|_{z=0} = -\frac{de^{-x^2/2}}{dx}, \quad (4.1.19)$$

and

$$G(x, z) = e^{\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} (e^{-x^2/2}) z^n = \sum_{n=0}^{\infty} h_n(x) \frac{z^n}{n!}. \quad (4.1.20)$$

Differentiating (4.1.16) with respect to  $x$ , we have

$$\frac{\partial G(x, z)}{\partial x} = zG(x, z) = \sum_{n=0}^{\infty} h'_n(x) \frac{z^n}{n!}. \quad (4.1.21)$$

Comparing the coefficients of  $z^n$  indicates that

$$h'_n(x) = nh_{n-1}(x). \quad (4.1.22)$$

With (4.1.22) and (4.1.22), the recursive relation of the un-normalized Hermite polynomials is obtained as

$$h_{n+1}(x) - xh_n(x) + nh_{n-1}(x) = 0, \quad n = 0, 1, 2, \dots \quad (4.1.23)$$

with  $h_{-1} = 0$  and  $h_0 = 1$ . Thus the first Hermite polynomials are

$$h_0(x) = 1$$

$$h_1(x) = x$$

$$h_2(x) = x^2 - 1$$

$$h_3(x) = x^3 - 3x$$

$$h_4(x) = x^4 - 6x^2 + 3$$

$$h_5(x) = x^5 - 10x^3 + 15x.$$

By (4.1.13), the recursive relation of normalized Hermite polynomials is

$$\sqrt{n+1}H_{n+1}(x) = xH_n(x) - \sqrt{n}H_{n-1}(x), \quad n = 0, 1, 2, \dots \quad (4.1.24)$$

and

$$H'_n(x) = \sqrt{n}H_{n-1}(x), \quad H_{-1}(x) = 0, \quad H_0(x) = 1. \quad (4.1.25)$$

The Hermite functions  $\xi_n(x)$  are defined by

$$\xi_n(x) = \pi^{-1/4} ((n-1)!)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}x), \quad n = 0, 1, 2, \dots \quad (4.1.26)$$

### 4.1.3 Wiener Chaos Expansion

Now let's define the tensor products

$$\xi_\delta := \delta_{(\delta_1, \dots, \delta_d)} := \xi_{\delta_1} \otimes \dots \otimes \xi_{\delta_d}, \quad (4.1.27)$$

where  $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$  denote  $d$ -dimensional multi-indices. The family of above tensor products forms an orthonormal basis for  $L^2(\mathbb{R}^d)$ . We will need to consider multi-indices of arbitrary length. To simplify the notation, we regard as elements of the space  $(\mathbb{N}_0^{\mathbb{N}})_c$  of all sequences  $\alpha = (\alpha_1, \alpha_2, \dots)$  with elements  $\alpha_i \in \mathbb{N}_0$  and with compact support, i.e., with only finitely many  $\alpha_i \neq 0$ . We write

$$\mathcal{J} = \left\{ \alpha = (\alpha_1, \alpha_2, \dots) : \alpha_i \in \mathbb{N}_0, |\alpha| := \sum_{i=1}^{\infty} \alpha_i < \infty \right\}, \quad (4.1.28)$$

and

$$\alpha! = \alpha_1! \alpha_2! \dots$$

$$\alpha < \beta \text{ if } \alpha_i < \beta_i \text{ for all } i \in \mathbb{N}$$

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots).$$

**Definition 1.** Let  $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$ , we define the Wick polynomial  $H_\alpha(\omega)$  of order  $|\alpha|$  as

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \xi_i \rangle), \quad \omega \in \mathcal{S}'(\mathbb{R}), \quad (4.1.29)$$

where  $\langle \omega, \xi_i \rangle$  denotes the standard Gaussian random variables. With  $\mu = \mu_m$ ,  $E = E_\mu$ ,  $\alpha = (\alpha_1, \alpha_2, \dots)$ ,  $\beta = (\beta_1, \beta_2, \dots)$ , we get

$$E[H_\alpha H_\beta] = \alpha! \delta_{\alpha\beta}, \quad (4.1.30)$$

and

$$E[H_0] = 1, \quad E[H_\alpha] = E[H_0 H_\alpha] = 0 \quad \text{if } \alpha \neq 0. \quad (4.1.31)$$

Then let's introduce the Cameron-Martin theorem [13].

**Theorem 4.1.2.** (*Cameron-Martin*) Every  $f \in L^2(\mu)$  has a unique representation

$$f(x, \omega) = \sum_{\alpha \in \mathcal{J}} f_\alpha(x) H_\alpha(\omega) \quad (4.1.32)$$

where  $f_\alpha$  denote the  $\alpha$ th Wiener chaos coefficients which are deterministic and  $H_\alpha$  are the random Wick polynomials defined by definition 4.1.29. Furthermore, the first two statistical moments of  $f$  are given by

$$E[f] = f_0 \quad \text{and} \quad E[f^2] = \sum_{\alpha \in \mathcal{J}} \alpha! |f_\alpha|^2. \quad (4.1.33)$$

Moreover, the variance of  $f(x, \omega)$  is given by

$$Var[f] = E[f^2] - (E[f])^2 = \sum_{\alpha \in \mathcal{J}, \alpha \neq 0} \alpha! |f_\alpha|^2 \quad (4.1.34)$$

## 4.2 Kondratiev Spaces and Wick Products

As we saw in the previous section, the growth condition

$$\sum_{\alpha} \alpha! f_\alpha^2 < \infty \quad (4.2.1)$$

assures that

$$f(x, \omega) = \sum_{\alpha \in \mathcal{J}} f_\alpha(x) H_\alpha(\omega) \in L^2(\mu) \quad (4.2.2)$$

**Definition 2.** (*The Kondratiev spaces of stochastic test function and stochastic distributions.*)  
[13]

a) *The stochastic test function spaces:* Let  $N$  be a natural number. For  $0 \leq \rho \leq 1$ , let

$$(S)_\rho^N = (S)_\rho^{m;N} \quad (4.2.3)$$

consist of those

$$f = \sum_{\alpha \in \mathcal{J}} f_\alpha(x) H_\alpha(\omega) \in L^2(\mu) \text{ with } f_\alpha \in \mathbb{R}^N \quad (4.2.4)$$

such that

$$\|f\|_{\rho,k}^2 := \sum_{\alpha \in \mathcal{J}} f_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all } k \in \mathbb{N} \quad (4.2.5)$$

where

$$f_\alpha^2 = |f_\alpha|^2 = \sum_{k=1}^N (f_\alpha^{(k)})^2 \text{ if } f_\alpha = (f_{\alpha_{pha}}^{(1)}, \dots, f_{\alpha_{pha}}^{(N)}) \in \mathbb{R}^N. \quad (4.2.6)$$

b) *The stochastic distribution spaces:*

For  $0 \leq \rho \leq 1$ , let

$$(S)_{-\rho}^N = ()_{-\rho}^{m;N} \quad (4.2.7)$$

consist of all formal expansions

$$f = \sum_{\alpha \in \mathcal{J}} f_{\alpha} H_{\alpha}(\omega) \quad \text{with } f_{\alpha} \in \mathbb{R}^N \quad (4.2.8)$$

such that

$$\|f\|_{-\rho, -q}^2 = \sum_{\alpha \in \mathcal{J}} f_{\alpha}^2 (\alpha!)^{1-\rho} (2N)^{-q\alpha} < \infty \quad \text{for some } q \in \mathbb{N}. \quad (4.2.9)$$

where  $(2N)^{k\alpha} = \prod_j (2j)^{k\alpha_j}$ .  $(S)_{\rho}^N$  and  $(S)_{-\rho}^N$  are called the Kondratiev spaces of stochastic test function and stochastic distributions [13], respectively.

**Remark 4.2.1.** Note that for general  $\rho \in [0, 1]$  we have

$$(S)_1^N \subset (S)_{\rho}^N \subset (S)_0^N \subset L^2(\mu) \subset (S)_{-0}^N \subset (S)_{-\rho}^N \subset (S)_{-1}^N \quad (4.2.10)$$

The Wick product was introduced in Wick (1950) as a tool to renormalize certain infinite quantities in quantum field theory. In stochastic analysis the Wick product was first introduced by Hida and Ikeda(1960). Today the Wick product is also important in the study of stochastic (ordinary and partial) differential equations. More details can be found in this book [13].

The (stochastic) Wick product can be defined in the following way:

**Definition 3.** The Wick product  $f \diamond g$  of two elements

$$f = \sum_{\alpha \in \mathcal{J}} f_{\alpha} H_{\alpha}, \quad g = \sum_{\beta \in \mathcal{J}} f_{\beta} H_{\beta} \in (S)_{-1}^{m;N} \quad \text{with } f_{\alpha}, g_{\beta} \in \mathbb{R}^N \quad (4.2.11)$$

is defined by

$$f \diamond g = \sum_{\alpha, \beta \in \mathcal{J}} f_{\alpha} g_{\beta} H_{\alpha+\beta}. \quad (4.2.12)$$

Furthermore, if  $H_{\alpha}$  is a orthonormal basis, the Wick product can also be rewritten as

$$f \diamond g = \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} f_{\alpha} g_{\beta} H_{\alpha+\beta}. \quad (4.2.13)$$

By replacing conditions (4.2.5) and (4.2.9) by  $\sup_{\alpha} \{f_{\alpha}^2 \alpha! (2N)^{k\alpha}\} < \infty$  for all  $k < \infty$  and  $\sup_{\alpha} \{f_{\alpha}^2 \alpha! (2N)^{-q\alpha}\} < \infty$  for some  $q < \infty$  respectively, we can define two other probability

spaces which are called the Hida test functions space  $(S)^N$  and the Hida distribution space  $(S)^{*,N}$  respectively. And the corresponding definite of Wick product is the same as above Wick definition. Then we have

$$(S)^N = (S)_0^N \text{ and } (S)^{*,N} = (S)_{-0}^N. \quad (4.2.14)$$

An important property of the spaces  $(S)_{-1}, (S)_1$  and  $(S)^*, (S)$  is that they are closed under Wick product.

**Lemma 4.2.1.** [13] For Kondratiev and Hida spaces and distribution sapces, we have

- a) If  $f, g \in (S)_{-1}^{m;N}$ , then  $f \diamond g \in (S)_{-1}^{m;1}$ .
- b) If  $f, g \in (S)_1^{m;N}$ , then  $f \diamond g \in (S)_1^{m;1}$ .
- c) If  $f, g \in (S)^{*,N}$ , then  $f \diamond g \in (S)^{*,1}$ .
- d) If  $f, g \in (S)^N$ , then  $f \diamond g \in (S)$ .

From the definition of Wick product, we can get the following basic algebraic properties.

**Lemma 4.2.2.** [13]

- a) (*Commutative law*) If  $f, g \in (S)_{-1}^{m;N}$ , then  $f \diamond g = g \diamond f$ .
- b) (*Associative law*) If  $f, g, h \in (S)_{-1}^{m;N}$ , then  $f \diamond (g \diamond h) = (f \diamond g) \diamond h$
- c) (*Distribution law*) If  $f, g, h \in (S)_{-1}^{m;N}$ , then  $f \diamond (g + h) = f \diamond g + f \diamond h$

**Definition 4. (Generalized expectation)** Let  $X = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (S)_{-1}^N$ . Then the vector  $c_0 = \tilde{X}(0) \in \mathbb{R}^N$  is called the generalized expectation of  $X$  and is denoted by  $E[X]$ . In the case when  $X = f \in L^p(\mu)$  for some  $p > 1$  then then generalized expectation of  $F$  coincides with the usual expectation

$$E[f] = \int_{S'} f(\omega) d\mu(\omega). \quad (4.2.15)$$

Note that  $E[f] = \langle f, 1 \rangle = f_0$  and

$$E[f \diamond g] = \langle E[f], E[g] \rangle, \text{ for all } f, g \in (S)_{-1}^N. \quad (4.2.16)$$

In particular,

$$E[f \diamond g] = E[f]E[g]; f, g \in (S)_{-1}. \quad (4.2.17)$$

### 4.3 Optimal control of stochastic elliptic equation

Now let's consider the Wiener Chaos solution of the stochastic elliptic equation (SEE)

$$\begin{cases} -(a_\epsilon(x, \omega) \diamond u'_\epsilon(x, \omega))' = f(x) + \theta(x) & x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0, \end{cases} \quad (4.3.1)$$

and define the objective function

$$J_\epsilon(u, \theta) = \frac{c}{2} E \left[ \int_0^1 |u_\epsilon - U|^2 dx \right] + \frac{1}{2} \int_0^1 \theta^2(x) dx, \quad (4.3.2)$$

where  $c$  is a constant,  $U(x)$  is deterministic and  $f(x) \in L^2(0, 1)$ .

**Remark 4.3.1.** Since the target  $U$  is deterministic, the above objective function can be rewritten as

$$J_\epsilon(u, \theta) = \frac{c}{2} \int_0^1 |u_0 - U|^2 dx + \frac{c}{2} \sum_{|\alpha| \geq 1, \alpha \in \mathcal{J}} \int_0^1 |u_\alpha|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx \quad (4.3.3)$$

$$= \frac{c}{2} \|u_0 - U\|_{L^2(Q_T)}^2 + \frac{c}{2} \|Var[u]\|_{L^2(Q_T)}^2 + \frac{1}{2} \|\theta\|_{L^2(Q_T)}^2. \quad (4.3.4)$$

In realistic models, the source of randomness can be expressed by a finite number of random variables that are mutually uncorrelated or mutually independent. For that reason, we assume that

$$a_\epsilon(x, \omega) = \sum_{n=0}^N a_{n,\epsilon}(x) \psi_n(\omega), \quad (4.3.5)$$

where  $\{\psi_i(\omega)\}$  is a series of random functions, which denotes the random sources.

Without loss of generality, we can always assume there exists a WCE of  $a_\epsilon(x, \omega)$ , which is

$$a_\epsilon(x, \omega) = \sum_{\alpha \in \mathcal{J}_a} a_{\alpha,\epsilon}(x) H_\alpha(\omega), \quad (4.3.6)$$

where  $\mathcal{J}_a$  is the set of indices of finite random sources.

**Theorem 4.3.1.** Let  $u(x, \omega) = \sum_{\beta \in \mathcal{J}} u_\beta(x) H_\beta(\omega)$  be a solution of equation (4.3.1), then the WCE coefficient  $u_\beta$  satisfies

a) If  $|\alpha| = |\beta| = 0$ , then

$$-(a_{\alpha,\epsilon}(x) u'_\beta(x))' = f(x) + \theta(x). \quad (4.3.7)$$



b) If  $|\alpha| + |\beta| \geq 1$ , then let  $\gamma = \alpha + \beta$  and for all  $\gamma \in \mathcal{J}$  and  $|\gamma| \geq 1$ ,

$$- \sum_{\substack{\beta \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) u'_\beta(x))' = 0. \quad (4.3.8)$$

**Proof:** Plug in WCEs of  $u(x, \omega)$  and  $a_\epsilon(x, \omega)$  to the equation (4.3.1), by (4.2.13) we have

$$- \left( \sum_{\beta \in \mathcal{J}} \sum_{\alpha \in \mathcal{J}_a} \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} a_{\alpha, \epsilon}(x) u'_\beta(x) H_{\alpha + \beta}(\omega) \right)' = f(x) + \theta(x). \quad (4.3.9)$$

I. If  $|\alpha| + |\beta| = 0$ , by  $H_0(\omega) = 1$ , the following equation can be derived from (4.3.9)

$$- (a_{\alpha, \epsilon}(x) u'_\beta(x))' = f(x) + \theta(x). \quad (4.3.10)$$

II. If  $|\alpha| + |\beta| \geq 1$ , let  $\gamma = \alpha + \beta$  and for all  $\gamma \in \mathcal{J}$  and  $|\gamma| \geq 1$ , another equation can also be derived from (4.3.9), which is

$$- \sum_{\substack{\beta \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) u'_\beta(x))' = 0. \quad (4.3.11)$$

Therefore, by Theorem 4.3.1, the (4.3.1) and (4.3.2) will be rewritten as

$$\begin{cases} - (a_{\alpha, \epsilon}(x) u'_\beta(x))' = f(x) + \theta(x), & |\alpha| = |\beta| = 0 \\ - \sum_{\substack{\beta \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) u'_\beta(x))' = 0, & \gamma \in \mathcal{J} \text{ and } |\gamma| \geq 1 \end{cases} \quad (4.3.12)$$

and the objective function

$$J_\epsilon(u, \theta) = \frac{c}{2} \int_0^1 |u_0(x) - U(x)|^2 dx + \frac{c}{2} \sum_{|\beta| \geq 1, \beta \in \mathcal{J}} \int_0^1 |u_\beta(x)|^2 dx + \frac{1}{2} \int_0^1 \theta^2(x) dx. \quad (4.3.13)$$

For the Winer chaos coefficients equation system (4.3.12), it is a infinite differential equation system and has to be truncated for the numerical solution purpose. Assume that we are going to keep the terms till the  $N$ -th order truncation with  $K$  Gaussian random variables. Let

$$\mathcal{J}_{K, N} = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_K) : \alpha_i \in \mathbb{N}_0, |\alpha| \leq N\}. \quad (4.3.14)$$

Then the solution  $u(x, \omega)$  will be truncated as

$$u(x, \omega) = \sum_{\alpha \in \mathcal{J}_{K, N}} u_\alpha(x) H_\alpha(\omega). \quad (4.3.15)$$

The total number of terms involved in above truncations for  $u(x, \omega)$  is  $\sum_{n=0}^N \binom{K+n-1}{n} = \frac{(K+N)!}{K!N!}$ , which increases very fast as  $K$  and  $N$  increase. Therefore, we need to apply an extra 'truncation' to (4.3.15), called sparse truncation [14], [23], to reduce the number of terms in Wiener chaos expansion. Let

$$r = (r_1, r_2, \dots, r_K) \text{ with } N = r_1 \geq r_2 \geq \dots \geq r_K. \quad (4.3.16)$$

Define

$$\mathcal{J}_{K,N}^r = \{(\alpha_1, \dots, \alpha_K) | \alpha_i \leq r_i \text{ and } |\alpha| \leq N\}. \quad (4.3.17)$$

Thus, the (4.3.15) will become

$$u(x, \omega) = \sum_{\alpha \in \mathcal{J}_{K,N}^r} u_\alpha(x) H_\alpha(\omega). \quad (4.3.18)$$

We also assume that

$$a_\epsilon(x, \omega) = \sum_{\alpha \in \mathcal{J}_{K,N}^r} a_{\alpha, \epsilon}(x) H_\alpha(\omega), \quad (4.3.19)$$

Let's look at following example with  $K = 6, N = 4$ . With the truncation (4.3.15), there are total  $\binom{9}{4} = 210$  terms in the truncation, while there are only total 26 terms in the sparse truncation (4.3.18). Such reducing in number of coefficients will reduce the computation cost dramatically. Let's introduce the adjoint function  $v_\beta(x)$ ,  $\beta \in \mathcal{J}$  satisfying

$$v_\beta(0) = v_\beta(1) = 0. \quad (4.3.20)$$

The associated Lagrangian can be expressed as

$$L(u, f; v) = J_\epsilon(\theta) + \int_0^1 v_0(x) [- (a_{0,\epsilon}(x) u'_0(x))' - f(x) - \theta(x)] dx \quad (4.3.21)$$

$$+ \sum_{|\gamma| \geq 1} \int_0^1 v_\gamma(x) [- \sum_{\substack{\beta \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) u'_\beta(x))'] dx \quad (4.3.22)$$

$$= J_\epsilon(\theta) - \int_0^1 v_0(x) (f(x) + \theta(x)) dx \quad (4.3.23)$$

$$+ \sum_{\gamma \in \mathcal{J}} \int_0^1 [- \sum_{\substack{\beta \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) v'_\gamma(x))' u_\beta(x)] dx \quad (4.3.24)$$

Now by method of variation, we have

$$\delta L = \sum_{\beta \in \mathcal{J}} \frac{\partial L}{\partial u_\beta} \delta u_\beta + \frac{\partial L}{\partial \theta} \delta \theta \quad (4.3.25)$$

$$= \int_0^1 \left( c(u_0(x) - U(x)) - \sum_{\gamma \in \mathcal{J}_a} (a_{\gamma, \epsilon}(x) v'_\gamma(x))' \right) \delta u_0(x) dx \quad (4.3.26)$$

$$+ \sum_{\beta \in \mathcal{J}} \int_0^1 \left( cu_\beta(x) - \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) v'_\gamma(x))' \right) \delta u_\beta(x) dx \quad (4.3.27)$$

$$+ \int_0^1 (\theta(x) - v_0(x)) \delta \theta dx. \quad (4.3.28)$$

Thus if letting

$$c(u_\beta(x) - U(x)) = \sum_{\gamma \in \mathcal{J}_a} (a_{\gamma, \epsilon}(x) v'_\gamma(x))', \quad \text{for } |\beta| = 0 \quad (4.3.29)$$

$$cu_\beta(x) = \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) v'_\gamma(x))', \quad \text{for } |\beta| \geq 1 \quad (4.3.30)$$

the variation of  $L$  is simplified as

$$\delta L = \int_0^1 (\theta(x) - v_\beta(x)) \delta \theta dx, \quad \text{where } |\beta| = 0. \quad (4.3.31)$$

This gives us a descent direction to update the control term  $\theta(x)$ ,

$$\delta \theta = -\lambda(\theta(x) - v_\beta(x)), \quad \text{where } |\beta| = 0 \quad (4.3.32)$$

where  $\lambda$  is the step size.

Therefore, the stochastic problem (4.3.12) and (4.3.13) are transformed to the equivalent system of equations (4.3.33).

$$\left\{ \begin{array}{l} -(a_{\beta, \epsilon}(x) u'_{\beta, \epsilon}(x))' = f(x) + \theta(x), \text{ for } |\beta| = 0 \\ - \sum_{\substack{\gamma \in \mathcal{J} \\ \beta - \gamma \in \mathcal{J}_a}} \sqrt{\frac{\beta!}{\gamma!(\beta - \gamma)!}} (a_{\beta - \gamma, \epsilon}(x) u'_{\gamma, \epsilon}(x))' = 0, \text{ for } |\beta| \geq 1 \\ c(u_{\beta, \epsilon}(x) - U(x)) - \sum_{\gamma \in \mathcal{J}_a} (a_{\gamma, \epsilon}(x) v'_\gamma(x))' = 0, \text{ for } |\beta| = 0 \\ cu_{\beta, \epsilon}(x) - \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (a_{\gamma - \beta, \epsilon}(x) v'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\ u_{\beta, \epsilon}(x) = v_\beta(x) = 0, \text{ for } x = 0, 1 \end{array} \right. \quad (4.3.33)$$

From Theorem 4.1.2 and (4.2.1), we know that there exist  $m_\beta$  and  $M_\beta$ , such that

$$0 < m_\beta \leq a_{\beta,\epsilon}(x) \leq M_\beta < \infty, \quad \forall \beta \in \mathcal{J} \text{ and } x \in (0, 1). \quad (4.3.34)$$

Then assume that the function  $\bar{a}_\beta$  is such that

$$\frac{1}{a_{\beta,\epsilon}} \rightharpoonup \frac{1}{\bar{a}_\beta}, \quad \text{weakly } * \text{ in } L^\infty(0, 1). \quad (4.3.35)$$

From the first equation of (4.3.33) and by the Theorem 2.1 in [18], for  $\beta \in \mathcal{J}$ , and  $|\beta| = 0$  we obtain that

$$u_{\beta,\epsilon} \rightharpoonup \bar{u}_\beta, \quad \text{as } \epsilon \rightarrow 0 \text{ weakly in } H_0^1(0, 1), \quad (4.3.36)$$

and  $\bar{u}_\beta$ , where  $|\beta| = 0$ , satisfies

$$-\frac{d}{dx} \left( \bar{a}_\beta(x) \frac{d\bar{u}_\beta}{dx} \right)' = f(x) + \theta(x), \quad |\beta| = 0 \text{ and } x \in (0, 1). \quad (4.3.37)$$

Then let us look at the second equation of (4.3.33) and assume that  $|\beta| = 1$ . Without loss of generality, let  $\beta = (1, 0, \dots, 0)$  and  $\beta_0 = \{0, 0, \dots, 0\}$ . We have

$$-(a_{\beta_0,\epsilon} u'_{\beta,\epsilon})' - (a_{\beta,\epsilon} u'_{\beta_0,\epsilon})' = 0, \quad (4.3.38)$$

i.e.

$$(a_{\beta_0,\epsilon} u'_{\beta,\epsilon} + a_{\beta,\epsilon} u'_{\beta_0,\epsilon})' = 0. \quad (4.3.39)$$

By Theorem 2.1 in [18], for  $u_{\beta,\epsilon}, \beta = (1, 0, \dots, 0) \in \mathcal{J}$  in (4.3.39) we have

$$u_{\beta,\epsilon} \rightharpoonup \bar{u}_\beta \quad \text{weakly in } H_0^1(0, 1), \quad (4.3.40)$$

where  $\bar{u}_\beta$  satisfies

$$(\bar{a}_{\beta_0} \bar{u}'_\beta + \bar{a}_{\beta_0\beta} \bar{u}'_{\beta_0})' = 0, \quad (4.3.41)$$

where  $\bar{a}_\beta = \frac{\bar{a}_{\beta_0}}{g_0}$  and  $g_0$  is such a function that

$$\frac{1}{g_\epsilon} = \frac{a_{\beta,\epsilon}}{a_{\beta_0,\epsilon}^2} \rightharpoonup \frac{1}{g_0}, \quad \text{weakly } * \text{ in } L^\infty(0, 1). \quad (4.3.42)$$

Similarly, for any  $\beta \in \mathcal{J}$ , by the first two equations of (4.3.33) we can obtain that there exist  $\bar{a}_\beta$  such that

$$u_{\beta,\epsilon} \rightharpoonup \bar{u}_\beta \quad \text{weakly in } H_0^1(0, 1), \quad (4.3.43)$$

where  $\{\bar{u}_\beta : \beta \in \mathcal{J}\}$  satisfy

$$\left\{ \begin{array}{l} -(\bar{a}_\beta(x)\bar{u}'_\beta(x))' = f(x) + \theta(x), \text{ for } |\beta| = 0 \\ -\sum_{\substack{\beta \in \mathcal{J} \\ \beta - \gamma \in \mathcal{J}_a}} \sqrt{\frac{\beta!}{\gamma!(\beta - \gamma)!}} (\bar{a}_{\beta - \gamma}(x)\bar{u}'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\ c(\bar{u}_\beta(x) - U(x)) - \sum_{\gamma \in \mathcal{J}_a} (\bar{a}_\gamma(x)v'_\gamma(x))' = 0, \text{ for } |\beta| = 0 \\ c\bar{u}_\beta(x) - \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (\bar{a}_{\gamma - \beta}(x)v'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\ \bar{u}_\beta(x) = v_\beta(x) = 0, \text{ for } x = 0, 1 \end{array} \right. \quad (4.3.44)$$

For general case, there is no explicit form for  $\bar{a}_\beta$ , except  $a_{\beta, \epsilon}$  is periodic.

Next, let us look at the special case and assume that

$$a_\epsilon(x, \omega) \text{ is periodic in terms of the first variable.} \quad (4.3.45)$$

Therefore, each of the coefficients  $a_{\alpha, \epsilon}(x)$  is also periodic in  $x$ . Based on the convergence results disussed above and [18], the solution  $u_{\beta, \epsilon}(x)$  of system (4.3.33) converges to the solution  $\bar{u}_\beta(x)$  of the following system (4.3.46) weakly.

$$\left\{ \begin{array}{l} -(\bar{a}_\beta(x)\bar{u}'_\beta(x))' = f(x) + \theta(x), \text{ for } |\beta| = 0 \\ -\sum_{\substack{\beta \in \mathcal{J} \\ \beta - \gamma \in \mathcal{J}_a}} \sqrt{\frac{\beta!}{\gamma!(\beta - \gamma)!}} (\bar{a}_{\beta - \gamma}(x)\bar{u}'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\ c(\bar{u}_\beta(x) - U(x)) - \sum_{\gamma \in \mathcal{J}_a} (\bar{a}_\gamma(x)v'_\gamma(x))' = 0, \text{ for } |\beta| = 0 \\ c\bar{u}_\beta(x) - \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (\bar{a}_{\gamma - \beta}(x)v'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\ \bar{u}_\beta(x) = v_\beta(x) = 0, \text{ for } x = 0, 1 \end{array} \right. \quad (4.3.46)$$

where

$$\bar{a}_\beta(x) = \left[ m \left( \frac{1}{a_\beta} \right) \right]^{-1}, \quad \beta = (0, 0, \dots, 0), \quad (4.3.47)$$

$$\bar{a}_\beta(x) = \frac{\bar{a}_{\beta_0}^2}{g_0}, \quad \beta_0 = (0, 0, \dots, 0) \text{ and } \beta = (1, 0, \dots, 0), \quad (4.3.48)$$

$$g_0 = \left[ m \left( \frac{a_\beta}{\bar{a}_{\beta_0}^2} \right) \right]^{-1}, \quad (4.3.49)$$

$$\dots \quad (4.3.50)$$

and

$$m(f) = \int_0^1 f(x) dx. \quad (4.3.51)$$

## 4.4 Numerical Example

### 4.4.1 Discretization of the control problem

Let us consider the following stochastic optimal control problem

$$\begin{cases} -(a_\epsilon(x, \omega) \diamond u'_\epsilon(x, \omega))' = f(x) + \theta(x) & x \in L = (0, 1), \\ u_\epsilon(0, \omega) = u_\epsilon(1, \omega) = 0, \end{cases} \quad (4.4.1)$$

with the objective function

$$J_\epsilon(u, \theta) = \frac{c}{2} E \left[ \int_0^1 |u_\epsilon - U|^2 dx \right] + \frac{1}{2} \int_0^1 \theta^2(x) dx. \quad (4.4.2)$$

Based on the discussion in the previous section, the optimal control problem (4.4.1) - (4.4.2)

will be transformed to

$$\left\{ \begin{array}{l}
- (\bar{a}_\beta(x) \bar{u}'_\beta(x))' = f(x) + \theta(x), \text{ for } |\beta| = 0 \\
- \sum_{\substack{\beta \in \mathcal{J} \\ \beta - \gamma \in \mathcal{J}_a}} \sqrt{\frac{\beta!}{\gamma!(\beta - \gamma)!}} (\bar{a}_{\beta - \gamma}(x) \bar{u}'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\
c(\bar{u}_\beta(x) - U(x)) - \sum_{\gamma \in \mathcal{J}_a} (\bar{a}_\gamma(x) v'_\gamma(x))' = 0, \text{ for } |\beta| = 0 \\
c\bar{u}_\beta(x) - \sum_{\substack{\gamma \in \mathcal{J} \\ \gamma - \beta \in \mathcal{J}_a}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} (\bar{a}_{\gamma - \beta}(x) v'_\gamma(x))' = 0, \text{ for } |\beta| \geq 1 \\
\bar{u}_\beta(x) = v_\beta(x) = 0, \text{ for } x = 0, 1 \\
\delta\theta = -\lambda(\theta(x) - v_\beta(x)), \text{ for } |\beta| = 0.
\end{array} \right. \quad (4.4.3)$$

Let us partition the interval  $[0, 1]$  into  $M$  subintervals with equal length  $\Delta x = \frac{1}{M}$ . Sparse truncation for WCE will also be applied to discretize the objective function, which is given by

$$J_{(K,N),M}^r = \frac{c\Delta x}{2} \sum_{i=0}^M |\bar{u}_{0,i} - U_i|^2 + \frac{c\Delta x}{2} \sum_{\beta \in \mathcal{J}_{K,N}^r, |\beta| \geq 1} |\bar{u}_{\beta,i}|^2 + \frac{\Delta x}{2} \sum_{i=0}^M \theta_i^2, \quad (4.4.4)$$

where  $\bar{u}_{\beta,i} = \bar{u}_\beta(x_i)$ . Denote the discretization of the first order and the second order derivatives by

$$u'(x_i) = \frac{(u_{i+1} - u_{i-1}))}{2\Delta x} = \frac{\delta u_i}{2\Delta x}, \quad u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = \frac{\delta^2 u_i}{\Delta x^2} \quad (4.4.5)$$

The discretization of (4.4.3) and boundary conditions at  $x_i = i\Delta x, i = 1, 2, \dots, M-1$ , will be

$$\left\{ \begin{array}{l} - \left( \frac{\delta \bar{a}_{\beta,i} \delta \bar{u}_{\beta,i}}{4\Delta x^2} + \frac{\bar{a}_{\beta,i} \delta^2 u_{\beta,i}}{\Delta x^2} \right) = f_i + \theta_i, |\beta| = 0 \\ - \sum_{\substack{\gamma \in \mathcal{J}_{K,N}^r \\ \beta - \gamma \in \mathcal{J}_{K,N}^r}} \sqrt{\frac{\beta!}{\gamma!(\beta - \gamma)!}} \left( \frac{\delta \bar{a}_{\beta-\gamma,i} \delta \bar{u}_{\gamma,i}}{4\Delta x^2} + \frac{\bar{a}_{\beta-\gamma,i} \delta^2 u_{\gamma,i}}{\Delta x^2} \right) = 0, |\beta| \geq 1 \\ c(\bar{u}_{\beta,i} - U_i) - \sum_{\gamma \in \mathcal{J}_{K,N}^r} \left( \frac{\delta \bar{a}_{\gamma,i} \delta v_{\gamma,i}}{4\Delta x^2} + \frac{\bar{a}_{\gamma,i} \delta^2 v_{\gamma,i}}{\Delta x^2} \right) = 0, |\beta| = 0 \\ c\bar{u}_{\beta,i} - \sum_{\substack{\gamma \in \mathcal{J}_{K,N}^r \\ \gamma - \beta \in \mathcal{J}_{K,N}^r}} \sqrt{\frac{\gamma!}{\beta!(\gamma - \beta)!}} \left( \frac{\delta \bar{a}_{\gamma-\beta,i} \delta v_{\gamma,i}}{4\Delta x^2} + \frac{\bar{a}_{\gamma-\beta,i} \delta^2 v_{\gamma,i}}{\Delta x^2} \right) = 0, |\beta| \geq 1 \\ \bar{u}_{\beta,0} = \bar{u}_{\beta,M} = 0, \beta \in \mathcal{J}_{K,N}^r \\ v_{\beta,0} = v_{\beta,M} = 0, \beta \in \mathcal{J}_{K,N}^r \\ \delta \theta_i = -\lambda(\theta_i - v_{\beta,i}), \quad |\beta| = 0. \end{array} \right. \quad (4.4.6)$$

Based on the discretized system (4.4.6), here are the iterative algorithm for this optimization:

**Algorithm:**

- Step 1. Initialize the control term  $\theta_i, i = 0, 1, \dots, M$ , and  $\lambda$  which is a small number.
- Step 2. Solve for  $(\bar{u}_{\alpha,i}, v_{\alpha,i})$  from (4.4.6)
- Step 3. Evaluate  $J_0 = J_{(K,N),M}^r(\bar{u}_{\alpha,i}, \theta)$
- Step 4. Set  $\theta_i = \theta_i - \lambda(\theta_i - v_{\alpha,i})$ , where  $|\alpha| = 0$
- Step 5. Solve for  $(\bar{u}_{\alpha,i}, v_{\alpha,i})$  from (4.4.6) with the new control term  $\theta$
- Step 6. Evaluate  $J_1 = J_{(K,N),M}^r(\bar{u}_{\alpha,i}, \theta)$
- Step 7. If  $J_1 > J_0$ , set  $\lambda = \frac{1}{2}\lambda$ , and go back to step 4. Otherwise, continue with next step.
- Step 8. If  $\frac{\|J_1 - J_0\|}{\|J_1\|} > \tau$ , where  $\tau$  is a tolerance, set  $\lambda = 1.5\lambda, J_0 = J_1$  and go back to step 4.

Otherwise stop.



#### 4.4.2 Numerical example

Let

$$a(x, \omega) = \frac{1}{1 + \cos(2\pi x)} + \frac{1}{1 - \sin(2\pi x)} \omega \quad (4.4.7)$$

where  $\omega$  is the standard Gaussian random variable and

$$\begin{aligned} a_\epsilon(x, \omega) &= a\left(\frac{x}{\epsilon}, \omega\right) = \frac{1}{1 + \cos(2\pi x/\epsilon)} + \frac{1}{1 - \sin(2\pi x/\epsilon)} \omega \\ &= \sum_{\alpha \in \mathcal{J}_{K,N}^r} a_{\alpha,\epsilon}(x) H_\alpha(\omega), \end{aligned} \quad (4.4.8)$$

where the coefficients  $a_{\alpha,\epsilon}$  are

$$a_{\alpha,\epsilon} = \frac{1}{1 + 0.5 \cos(2\pi x/\epsilon)}, \quad \text{if } \alpha = (0, 0, \dots, 0) \quad (4.4.9)$$

$$a_{\alpha,\epsilon} = \frac{1}{1 - 0.5 \sin(2\pi x/\epsilon)}, \quad \text{if } \alpha = (1, 0, \dots, 0) \quad (4.4.10)$$

$$a_{\alpha,\epsilon} = 0, \quad \text{if } \alpha \in \mathcal{J}_{K,N}^r \text{ and } \alpha \neq (0, 0, \dots, 0) \text{ or } (1, 0, \dots, 0). \quad (4.4.11)$$

Therefore, by (4.3.47) - (4.3.49) one can obtain that,

I for  $|\alpha| = 0$ ,

$$\bar{a}_\alpha = \left[ \int_0^1 (1 + 0.5 \cos(2\pi x)) dx \right]^{-1} = 1; \quad (4.4.12)$$

II for  $|\alpha| = 1$  and  $\alpha_1 = 1$ ,

$$\bar{a}_\alpha = \frac{1}{\left[ \int_0^1 \frac{(1 + 0.5 \cos(2\pi x))^2}{1 - 0.5 \sin(2\pi x)} dx \right]^{-1}} = \int_0^1 \frac{(1 + 0.5 \cos(2\pi x))^2}{1 - 0.5 \sin(2\pi x)} dx = 1.2887. \quad (4.4.13)$$

Also assume that

$$U(x) = \sin(2\pi x), \text{ and } f(x) = x^2.$$

With above assumptions, (4.4.6) will become

$$\left\{ \begin{array}{l} -\frac{\bar{a}_{\beta,i}\delta^2 u_{\beta,i}}{\Delta x^2} = f_i + \theta_i, \text{ for } |\beta| = 0 \\ -\sum_{\substack{\gamma \in \mathcal{J}_{K,N}^r \\ \beta-\gamma \in \mathcal{J}_{K,N}^r}} \sqrt{\frac{\beta!}{\gamma!(\beta-\gamma)!}} \frac{\bar{a}_{\beta-\gamma,i}\delta^2 u_{\gamma,i}}{\Delta x^2} = 0, \text{ for } |\beta| \geq 1 \\ c(\bar{u}_{\beta,i} - U_i) - \sum_{\gamma \in \mathcal{J}_{K,N}^r} \frac{\bar{a}_{\gamma,i}\delta^2 v_{\gamma,i}}{\Delta x^2} = 0, \text{ for } |\beta| = 0 \\ c\bar{u}_{\beta,i} - \sum_{\substack{\gamma \in \mathcal{J}_{K,N}^r \\ \gamma-\beta \in \mathcal{J}_{K,N}^r}} \sqrt{\frac{\gamma!}{\beta!(\gamma-\beta)!}} \frac{\bar{a}_{\gamma-\beta,i}\delta^2 v_{\gamma,i}}{\Delta x^2} = 0, \text{ for } |\beta| \geq 1 \\ \bar{u}_{\beta,0} = \bar{u}_{\beta,M} = 0, \text{ for } \beta \in \mathcal{J}_{K,N}^r \\ v_{\beta,0} = v_{\beta,M} = 0, \text{ for } \beta \in \mathcal{J}_{K,N}^r. \end{array} \right. \quad (4.4.14)$$

First, let's look at the oscillation of coefficient  $a_\epsilon(x, \omega)$  for different small values of  $\epsilon$ . Figure

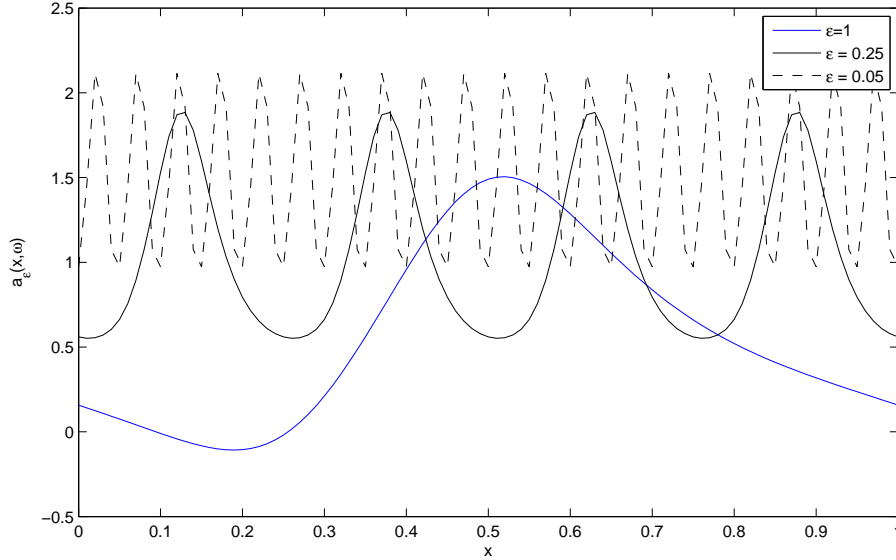


Figure 4.1 The graph of  $a_\epsilon(x, \omega)$  with  $\epsilon = 1, \epsilon = 0.25$  and  $\epsilon = 0.05$ . The smaller the  $\epsilon$  is, the more oscillated the coefficient is.

4.1 shows that the coefficients is oscillated for small  $\epsilon$  and there may be difficulty if solving directly. Therefore, homogenization method will avoid solving systems with  $\epsilon$  involved. We choose tolerance to be  $1.0 \times 10^{-5}$  and the initial step size to be 0.01. With the iterative

Table 4.1 Values of objective function  $J$  for each iteration

Iteration	Value of J	Iteration	Value of J	Iteration	Value of J
1	89.2031669221943	11	26.2170293582106	21	26.1745693544463
2	33.1270823149303	12	26.2051433830445	22	26.1739820059215
3	28.0464137644130	13	26.1965463199116	23	26.1735422976979
4	26.9684256252617	14	26.1902665511716	24	26.1732129668261
5	26.6022888514217	15	26.1856470717718	25	26.1729662213385
6	26.4365476226976	16	26.1822316037798	26	26.1727813040845
7	26.3468841778225	17	26.1796969609125		
8	26.2926813669778	18	26.1778108615513		
9	26.2575360548715	19	26.1764045422710		
10	26.2336940063908	20	26.1753543980168		

algorithm mentioned above, we have the values of objective function for each iteration shown in Table 4.1

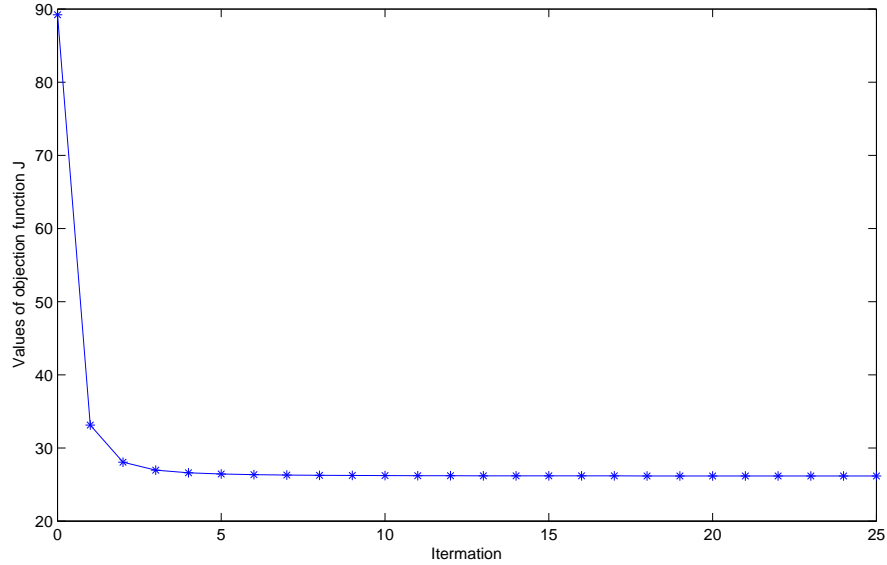
Figure 4.2 The graph of the convergence of values of objective function  $J$ .

Table 4.1 and Figure 4.2 show that the values of objective function converges to a value, which can be approximated as the minimization of the original objective function, which is 26.1727813040845. The summary is shown in Table 4.2.

Next, let's look at the graphs of  $u(x, \omega)$ ,  $E[u(x, \omega)]$  and  $U(x)$ , as shown in Figure 4.3.

Table 4.2 Summary of objective function

Initial value of $J$	Final value of $J$	Total number of iterations
89.2031669221943	26.1727813040845	25

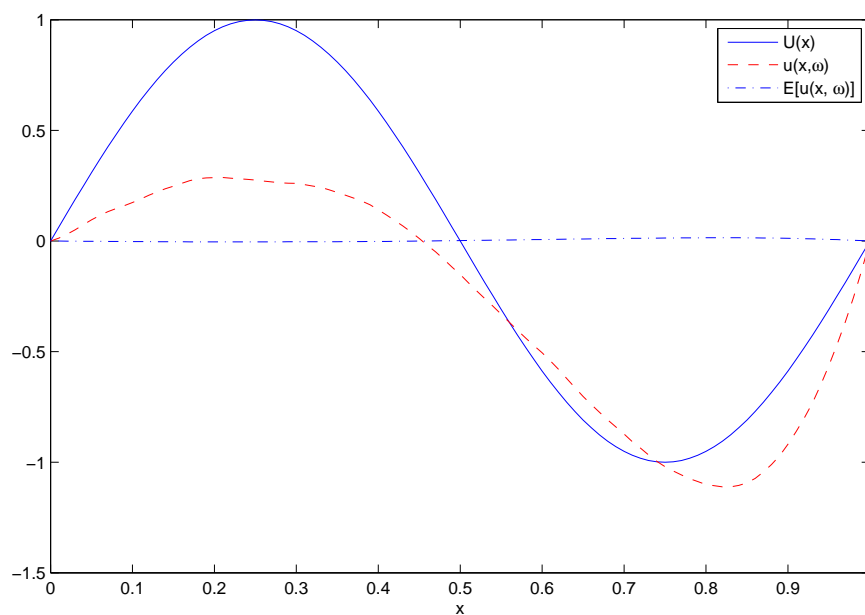


Figure 4.3 The graph of WCE solution  $u(x, \omega)$  under control, its mean value  $E[u(x, \omega)]$  and the target function  $U(x)$ .

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